Electromagnetic waves in a plasma containing both electric charges and magnetic monopoles

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(Received 13 August 1981; accepted for publication 11 November 1981)

We study electromagnetic wave propagation in a hypothetical cold plasma containing both electric and magnetic charges. The waves exhibit rather unusual properties. In particular, they can propagate below the characteristic frequencies. Reflection on a half-space and electric dipolar radiation are discussed.

I. INTRODUCTION

Because of the duality of Maxwell's equations with respect to electric and magnetic quantities,[1] a plasma containing solely magnetic charges[2] would behave qualitatively as an ordinary plasma (exchanging the roles of electric and magnetic fields). So, we consider a plasma containing both electric and magnetic charges.

As could be expected, such a plasma exhibits rather unusual properties for wave propagation. In addition to its academic relevance for magnetic monopoles, this problem is of pedagogical value for teaching waves in plasmas: owing to the odd properties, erroneous results are obtained if usual "rules of thumb" are applied uncritically.

In Sec. II we derive the dispersion equation for electromagnetic waves; then we solve two classical problems: reflection on a plasma half-space (Sec. III) and radiation of a small electric dipole (Sec. IV).

II. DERIVATION OF THE DISPERSION EQUATION

A. Maxwell equations

If both electric and magnetic charges are present, Maxwell's equations have the following symmetric form:

$$\begin{aligned}\nabla\times\mathbf{E} &= -\mu_0\,\mathbf{J}^M - \partial\mathbf{B}/\partial t,\\ \nabla\times\mathbf{B} &= \mu_0\,\mathbf{J}^E + 1/c^2\,\partial\mathbf{E}/\partial t;\end{aligned}\tag{1}$$

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \rho^E/\epsilon_0,\\ \nabla\cdot\mathbf{B} &= \mu_0\rho^M,\end{aligned}\tag{2}$$

in SI units. Here $\rho^E(\rho^M)$ and $J^E(J^M)$ denote the electric (magnetic) charge and current densities. Implicit in these equations are the usual conservation relations

$$\begin{aligned}\nabla\cdot\mathbf{J}^E + \partial\rho^E/\partial t &= 0,\\ \nabla\cdot\mathbf{J}^M + \partial\rho^M/\partial t &= 0.\end{aligned}\tag{3}$$

To deduce the wave dispersion equation, we need the relations between the charge and current densities and the electromagnetic field.

B. Constitutive relations

The plasma is defined as containing (i) ordinary electric charges (n_E particles per unit volume, of charge e_E, mass m_E) and (ii) magnetic charges (n_M particles per unit volume, of magnetic charge e_M, mass m_M).

To ensure (electric and magnetic) charge neutrality, we introduce two more species, with equal densities, and charges opposite to the previous ones. The simplest hypothesis is to assume that the masses of these latter species are sufficiently large to render their movement in the fields negligible at the frequencies considered.

We assume that there are no static fields; in addition we use the cold plasma approximation, i.e., the particles are at rest at equilibrium. We also use a classical, nonrelativistic approximation, and linearize the equations.

Thus the electric (magnetic) charges velocities $\mathbf{v}^E$ ($\mathbf{v}^M$) satisfy the equations

$$m_E\,\partial\mathbf{v}^E/\partial t = e_E\mathbf{E},\quad m_M\,\partial\mathbf{v}^M/\partial t = e_M\mathbf{B},\tag{4}$$

where the terms $e_E\mathbf{v}^E\times\mathbf{B}$ ($e_M\mathbf{v}^M\times\mathbf{E}/c^2$) have been neglected in the linearization.

If there are no exterior sources, the total current densities are

$$\mathbf{J}^E = n_E e_E\mathbf{v}^E,\quad \mathbf{J}^M = n_M e_M\mathbf{v}^M.\tag{5}$$

C. Dispersion equation

As usual, we perform a space-time Fourier transform defined by

$$\varphi(\mathbf{k},\omega) = \int d^3r\,dt\,e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\,\varphi(\mathbf{r},t).$$

Equations (4) and (5) yield

$$\mathbf{J}^E = i\epsilon_0\mathbf{E}\omega_p^2/\omega,\quad \mathbf{J}^M = i(\mathbf{B}/\mu_0)\omega_M^2/\omega,\tag{6}$$

where we have defined the plasma frequencies

$$\omega_p = (n_E e_E^2/\epsilon_0\,m_E)^{1/2},\quad \omega_M = (\mu_0 n_M e_M^2/m_M)^{1/2}.$$

Substituting Eq. (6) into the transformed Eqs. (1)–(3) yields

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}\epsilon_M,\quad \mathbf{k}\times\mathbf{B} = -\omega\mathbf{E}\epsilon_p/c^2,\tag{7}$$

$$\epsilon_p\mathbf{k}\cdot\mathbf{E} = 0,\quad \epsilon_M\mathbf{k}\cdot\mathbf{B} = 0,\tag{8}$$

where $\epsilon_p = 1 - \omega_p^2/\omega^2$, $\epsilon_M = 1 - \omega_M^2/\omega^2$, and, for brevity, the symbols $\mathbf{E}$, ... stand for the Fourier transforms $\mathbf{E}(\mathbf{k},\omega)$.

Thus the fields satisfy the set of equations

$$\Lambda_{ij}(\mathbf{k},\omega)\,E_j(\mathbf{k},\omega) = 0\tag{9}$$

and a similar equation for $\mathbf{B}$, with

$$\Lambda_{ij}(\mathbf{k},\omega) = k_i k_j c^2/\omega^2 + [\epsilon(\omega) - k^2c^2/\omega^2]\delta_{ij},\tag{10}$$

where $\epsilon(\omega) = \epsilon_p\epsilon_M$.

The dispersion equation

$$\det(\Lambda_{ij}) = 0\tag{11}$$

factorizes into the following two equations:

(i) $\epsilon(\omega) = 0$, which splits in turn into two equations. There is a longitudinal electric field oscillation at $\omega = \omega_p$, with $\mathbf{k}\times\mathbf{E} = 0$, $\mathbf{B} = 0$. This is the analog of the ordinary plasma oscillation. There is also a longitudinal magnetic field oscillation at $\omega = \omega_M$ with $\mathbf{k}\times\mathbf{B} = 0$, $\mathbf{E} = 0$. In the special case where $\omega = \omega_p = \omega_M$, there is a longitudinal oscillation of both **E** and **B**. We do not comment further on these solutions; like the usual plasma waves,[3] they propagate if the plasma is warm.

(ii) $\epsilon(\omega) = k^2c^2/\omega^2$. This corresponds to transverse electromagnetic waves with $\mathbf{k}\cdot\mathbf{E} = \mathbf{k}\cdot\mathbf{B} = 0$.

D. Electromagnetic waves

The transverse electromagnetic waves have the dispersion equation

$$k^2c^2/\omega^2 = (1 - \omega_p^2/\omega^2)(1 - \omega_M^2/\omega^2), \tag{12}$$

which yields the phase velocity

$$v_\varphi = \omega/k = c[(1 - \omega_p^2/\omega^2)(1 - \omega_M^2/\omega^2)]^{-1/2} \tag{13}$$

and the group velocity

$$\begin{aligned} \mathbf{v}_g &= \partial\omega/\partial\mathbf{k} = \mathbf{k}c^2/[\omega(1 - \omega_p^2\omega_M^2/\omega^4)] \\ &= (\mathbf{k}/k)v_g. \end{aligned} \tag{14}$$

These waves propagate without attenuation either above both ω_p and ω_M, or below both ω_p and ω_M. In the special case where $\omega_p = \omega_M$, the waves can propagate at any frequency.

The high-frequency propagation can be understood heuristically as for a normal plasma: since charged particles with response time $1/\omega_p$ $(1/\omega_M)$ cannot keep up with the change in the electromagnetic field at the frequency ω, they do not compensate it exactly.

On the other hand, the low-frequency propagation appears more unusual. It may be understood from the fact that, while each charge species, if alone, could follow imposed low-frequency fields, so as to compensate them, this is not possible when both species are present, owing to the peculiar coupling displayed in Eqs. (7). In the low-frequency range, the waves have an unusual property: as shown in Eqs. (7), when both ϵ_p and ϵ_M are negative, the orthogonal set of vectors **E**, **B**, **k**, has a handedness that is opposite to the usual one. In other words $\mathbf{E}\times\mathbf{B}$ has a direction opposite to **k**. This is in agreement with the fact [shown in Eq. (14)] that in this frequency range, the group velocity has a direction opposite to **k**.

Similar unusual features sometimes appear in ordinary plasmas: for example, in a warm plasma with a static magnetic field, the group velocity of the plasma wave may make an obtuse angle with the wave vector **k**[4]; similar features appear in some crystalline media.[4] But, contrary to the present problem, such situations are generally associated with anisotropy and/or dissipative effects, yielding more complicated equations or concepts.

Sections III and IV show an application to two classical problems.

III. REFLECTION ON A PLASMA HALF-SPACE

Let a plane electromagnetic wave [wave vector $\mathbf{k}_0 = (\omega/c)\mathbf{e}_z$, where e_z is the unit vector in the **z** direction] be incident normally from vacuum on a half-space $(z > 0)$ of such a plasma, and let us calculate the reflection coefficient.

Applying Stoke's theorem to Eqs. (1) and (2) shows that the tangential components of **E** and **B** are continuous in the absence of surface currents, as in the usual case. On the other hand, the remaining boundary conditions, which may be deduced from Eq. (7), are different from the usual ones.

Owing to the symmetry of the problem, the wave vector **k** of the wave transmitted in the plasma is also parallel to *Oz*. Its direction is determined so as to ensure that the group velocity be in the **z** direction. Thus, using Eq. (12),

$$\mathbf{k} = \mathbf{k}_0\epsilon^{1/2}\,\mathrm{sgn}(1 - \omega_p^2\omega_M^2/\omega^4). \tag{15}$$

Thus, at low frequencies, **k** is in the direction opposite to $\mathbf{k}_0$.

The boundary conditions (continuity of **E** and **B**, which are tangential to the separation surface, and relations between **E** and **B** given by Eqs. (7) with, in vacuum, $\epsilon_p = \epsilon_M = 1$) yield

$$\begin{aligned} E_i + E_r &= E_t, \\ E_i - E_r &= (\mathbf{k}/\mathbf{k}_0)E_t/\epsilon_M, \end{aligned} \tag{16}$$

where the subscripts *i*, *r*, and *t* stand, respectively, for the incident, reflected, and transmitted quantities. Inserting Eq. (15) in Eqs. (16) gives the reflection coefficient

$$R = \left|\frac{E_r}{E_i}\right|^2 = \left|\frac{1 - (\epsilon_p/\epsilon_M)^{1/2}}{1 + (\epsilon_p/\epsilon_M)^{1/2}}\right|^2. \tag{17}$$

If $\omega_M = 0$, this reduces to the usual result.

Note, that setting carelessly **k** in the *Oz* sense as usual, makes $R > 1$ in the low-frequency range!

Equation (17) shows an interesting property: if $\omega_p = \omega_M$, then $R = 0$ for any ω; the electromagnetic wave is entirely transmitted into the plasma at all frequencies.

IV. ELECTRIC DIPOLAR RADIATION

We set an infinitesimal electric dipole in an infinite plasma of this type, and calculate the radiation resistance.

The dipole is defined by its current distribution

$$\mathbf{J}^a(\mathbf{r}) = \mathbf{e}_z 2I_0L\delta(\mathbf{r}).$$

Applying Parseval's theorem, the time-averaged power radiated by the electric current $\mathbf{J}^a e^{-i\omega t}$ is given by

$$\begin{aligned} \overline{P} &= -\tfrac{1}{2}\,\mathrm{Re}\int \mathbf{E}(\mathbf{r})\cdot\mathbf{J}^{a*}(\mathbf{r})d^3r \\ &= -\frac{1}{2(2\pi)^3}\,\mathrm{Re}\int \mathbf{E}(\mathbf{k})\cdot\mathbf{J}^{a*}(\mathbf{k})d^3k, \end{aligned} \tag{18}$$

where $\mathbf{E}(\mathbf{r})e^{-i\omega t}$ is the field in the presence of the electric current $\mathbf{J}^a(\mathbf{r})e^{-i\omega t}$, and $\mathbf{E}(\mathbf{k})$, $\mathbf{J}^a(\mathbf{k})$ denote the corresponding spatial Fourier transforms. Since $\mathbf{J}^a(\mathbf{k}) = 2\mathbf{z}I_0L$, this yields

$$\overline{P} = -\frac{I_0L}{(2\pi)^3}\,\mathrm{Re}\int E_z(\mathbf{k})d^3k. \tag{19}$$

ACKNOWLEDGMENT

This problem was suggested by Pascal P. Meyer.

[1]See a discussion in J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 251.

[2]P. A. M. Dirac, Proc. R. Soc. London A **133**, 60 (1931).

[3]See, for instance, T. H. Stix, *The theory of plasma waves* (McGraw-Hill, New York, 1962).

[4]V. L. Ginzburg, *The propagation of electromagnetic waves in plasmas* (Pergamon, New York, 1964), pp. 145 and 472.

[5]See, for instance, R. F. Harrington, *Time-harmonic electromagnetic fields* (McGraw-Hill, New York, 1961).

[6]See, for instance, H. H. Kuehl, Radio Sci. **1**, 971 (1966); N. Meyer-Ver-

net, Astron. Astrophys. **97**, 208 (1981).

[7]See, for instance, among recent textbooks: J. Lighthill, *Waves in Fluids* (Cambridge University, Cambridge, 1978), pp. 361–366; or E. G. Harris, *Modern theoretical physics* (Wiley, New York, 1975), Vol. I, pp. 217–220.

[8]In this frequency range, the phase velocity may be lower than c, so that a moving charged particle should emit electromagnetic waves by Cherenkov radiation; conversely, a finite temperature of the plasma should give rise to Landau damping[3] of electromagnetic waves.

Effect of boundary conditions on the behavior of Bloch electrons

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(Received 11 May 1981; accepted for publication 15 October 1981)

The behavior of Bloch electrons in a uniform electric field is considered from first principles with the result that the crucial significance of certain mathematical and conceptual difficulties, which have been the source of considerable debate in the literature, becomes apparent. For several typical configurations, it is shown that the choice of boundary conditions has a drastic effect on the density-of-states function $D(\epsilon)$. For certain configurations, $D(\epsilon)$ will execute an unacceptable and discontinuous behavior as the electric field is varied. Conversely, when compatible boundary conditions are used, $D(\epsilon)$ varies smoothly over all electric field values, converging uniformly as the field becomes vanishingly small. Special problems concerning the order of taking the limits $\mathscr{E}\to 0$, $L\to\infty$ are brought out. The energy eigenstates will not generally converge to the zero-field Bloch-type states in the limit as $\mathscr{E}\to 0$ (unless special precautions are taken).

I. INTRODUCTION

Students of solid-state physics should be thoroughly exposed to the fundamental mechanisms of electrical conduction in solids. Unfortunately, conduction involves so many complex processes that any direct approach at modelling this phenomenon is virtually impossible for the numerous configurations employed. Since closed-form solutions exist for only a few, simple macroscopic cases (i.e., Ohm's law) it is necessary for the student to explore various general methods of approximation that will yield tractable solutions. It is expedient, therefore, to develop approximate approaches that can reduce unmanageable problems to manageable, simple, special cases which should provide global insights into the processes of conduction and the limitations of the physical structures themselves.

Although the behavior of an electron moving under the simultaneous influence of a lattice potential and a uniform applied electric field has been studied extensively,[1–9] the subject is still controversial[10,11] even though five decades have now passed since Bloch first published his classical paper on the subject.[12] Major controversies include the following subjects: (i) the existence of Stark ladders, (ii) Bloch oscillations, and (iii) the validity of infinite versus bounded lattices.[4] Stark ladders have been postulated as quantized energy levels in solids with a spacing that should be proportional to a superimposed, constant electric field[13,14] similar to Landau level modification in a magnetic field. Bloch oscillations are thought to exist when an electron in a lattice with well-separated bands oscillates in a superimposed, constant electric field. The existence of such oscillations has been questioned.[15–17]

Ideally, one would hope to obtain an exact solution to Schrödinger's equation for the complete system of an electron that is free to move under the influence of a lattice potential and an applied electric field. It is not a trivial problem, however, being so complicated that one is forced to consider numerical solutions. Such solutions provide some perspective but they do not lead to the cause-and-effect insights that closed-form solutions might.

With the vast amount of controversy that exists, it is essential to identify certain critical factors. In this regard, we feel it is instructive to treat two physical situations: one in which the electric field is suppressed and the other in which the lattice potential is suppressed. Study of one or the other of these two cases yields considerable insight into the general problem. Since the null-field case has been amply treated in the literature, and since it is the presence of an electric field and its unique contribution to the problem that is responsible for much of the controversy, we choose to emphasize that aspect of the problem. We believe it is essential to concentrate on examples where the electric field dominates and the lattice is assumed to be null. (These cases are significant, for example, in the study of ultrathin semiconductor structures where surface boundary and field intensity effects are more dominant than the lattice potential.[18]

The crucial influence of boundary conditions will become obvious when viewed from the perspective of this fundamental approach. This has a direct bearing on the approach one should take in the more general situation where both the lattice potential and the impressed field act upon the electron. In fact, if this had previously been emphasized in introductory solid-state conduction, then perhaps the present dilemmas and controversies appearing in the major physics journals could have been avoided. To