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$$H = (\mathbf{P} - e\mathbf{A})^2/2M,$$

where $A$ is the vector potential associated with monopole. Comparing with the Berry Hamiltonian—Eq. (24)—it is clear that the *parameter* space field $\mathbf{B}(\mathbf{R})$ corresponds to the "*real world*" case $eg\mathbf{B}(\mathbf{r})$.
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# Nonradiating sources: The subtle art of changing light into black

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When a point charge accelerates or moves faster than light in a dielectric medium, it radiates. However, sources of finite size can be designed whose peculiar structure ensures that they do not radiate under these conditions. The criterion for absence of radiation of a rigid source in free space is generalized to a dielectric medium, and applied to either oscillating or Čerenkov sources.

## I. INTRODUCTION

"Let there be electricity and magnetism, and there is light!" says Feynman's personal version of Genesis[1]; and, indeed, when an electron is accelerated, it radiates. Though as human beings we appreciate this property, as physicists, we do not: Separating sense from nonsense in the equations of a radiating electron is an old dream and, as Einstein once said,the electron is a stranger in electrodynamics.[2]

Even the classical electron at rest is odd: Since like charges repel, the Coulomb field tends to make it explode (unless its "mechanical mass" is negative), and one must imagine rubber strips such as the "Poincaré stresses" to hold the charge together. Anyway, the electrostatic energy of a point charge is infinite.

If the classical point electron can accelerate, the situation grows worse: While radiating, the particle undergoes a radiation reaction that has two parts (in its rest frame). The first one, which is infinite, can be viewed as a contribution to the mass, since it goes as $d^2\mathbf{r}/dt^2$. But the second one cannot be so "renormalized" since it is proportional to $d^3\mathbf{r}/dt^3$, and can cause the charge to accelerate itself[3,4] (the so-called runaway solution).

Although quantum electrodynamics is renormalizable and powerful tools[5,6] have been devised to hide the infinities under the carpet, it nevertheless cannot yield a finite energy for the point charge, nor a satisfactory theory of an extended charge.

The old problem of building a clean model for the classical electron has motivated a search for charge distributions that do not radiate.[7] Such charges could undergo force-free accelerated motion.[8]

At present, the interest of such sources seems academic:

  

They must be rigid in one particular reference frame regardless of their motion, and thus not relativistically invariant. By contrast, their relativistic counterparts (namely, those that are rigid in their own rest frame) do radiate.[9] In addition, they have not been quantized. They are, however, interesting from a pedagogical point of view as a means of enlightening the physics of radiation.

How can one design source distributions that do not radiate? A simple criterion has been derived[10] for time-periodic sources in empty space, and generalized in the relativistic formalism.[9] But what happens in a dielectric medium? This brings about an important change: A source does not need to accelerate in order to radiate, if it moves faster than light in the medium. The question then arises of whether it is also possible to design sources that do not emit Čerenkov radiation.

The purpose of this article is to derive in a simple way a condition for absence of radiation in a lossless dielectric medium and to apply it to either an oscillating or a Čerenkov source. Since the underlying physics is closely connected to diffraction or antenna problems, we use the same formalism, i.e., Fourier transforms. In order to remain at a rather elementary level, we use the usual physicists' loose tricks for taming generalized functions; in addition, we do not use a relativistically invariant formalism and consider only nonrelativistic velocities. Units are SI.

## II. A SIMPLE LOOK AT NONRADIATING SOURCES

Imagine a point charge $q$ in a lossless, nondispersive, uniform, and time-independent dielectric: The permittivity is a real constant $\epsilon > 0$, and the light velocity $c' = c/n$ with $n = \epsilon^{1/2}$.

First, let the charge be oscillating at the angular frequency $\omega$. For small amplitude motion, this is equivalent to a pulsating source at $\omega$, which emits waves with wavenumber $k = \omega/c'$ (otherwise, one must consider a whole spectrum of $\omega$ and $k$).

Now, instead of oscillating, let the charge be moving with a constant velocity $\mathbf{v}$. Does it radiate?

If $v < c'$, one can formally transpose the free-space results, replacing $c$ by $c'$ (and $q$ by $q' = q/n$), and conclude that there is no radiation.

If $v > c'$, however, this conclusion does not hold. The situation is illustrated in Fig. 1. The disturbances initiated by the charge on each point of its trajectory arrive simultaneously at the surface of a cone of angle $\theta = \sin^{-1}(c'/v)$ trailing the particle (since the particle traverses AO in the same time that the light travels from A to M): One expects a forward radiation in the direction $\theta_0 = \pi/2 - \theta$.

The emission does not stem from the charge itself, but from the transitory dipoles induced in the medium near the particle track, which interfere constructively on the cone. It is, however, instructive to look at the result in terms of the Liénard–Wiechert potentials[11] of the particle, for example,

$$\Phi(\mathbf{r},t) = q'/[4\pi\epsilon_0(R - \mathbf{v}\cdot\mathbf{R}/c')],$$

with $\mathbf{R} = \mathbf{r} - \mathbf{v}t'$, $t' = t - R/c'$. For any observation point $\mathbf{r}$ inside the cone there are two solutions for the retarded time $t'$ since each point $\mathbf{r}$ (such as I) is on two equiphase circles. But if $\mathbf{r}$ is on the surface of the cone, then the two solutions $t'$ merge together, whence $R - \mathbf{v}\cdot\mathbf{R}/c' = 0$, causing the fields to be infinite.

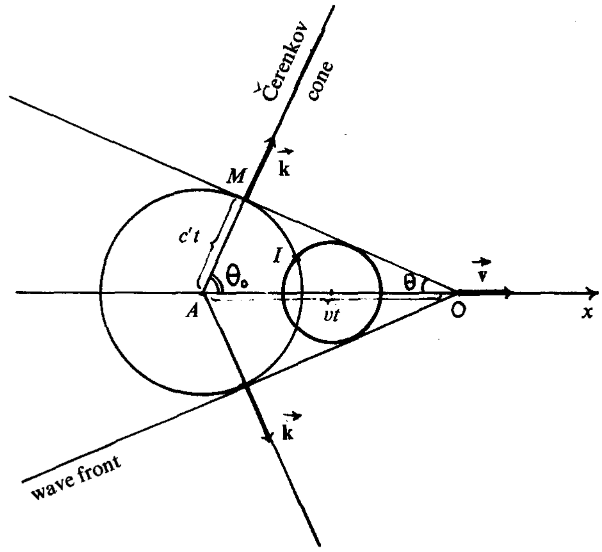


Fig. 1. Sketch of the Čerenkov emission of a point charge in a (nondispersive) dielectric medium.

What is wrong? We know that a strictly nondispersive dielectric does not exist: In particular, when the field varies, the polarization cannot follow immediately. This means that in the limit of short time scales (or high frequencies), $c'$ tends to $c$, so that the Čerenkov condition $v > c'$ no longer holds. The correct procedure is, therefore, to perform a calculation for each (angular) frequency $\omega$ and integrate over the *finite* frequency range defined by $vn(\omega) > c$.

The wave-front cone is analogous to that of a supersonic bullet in air, or to the wake of a boat in sufficiently shallow water; in deep water, however, dispersion is important since the phase velocity is $g/\omega$[12] ($g$ being the acceleration of gravity), so that the boat velocity is always larger than the phase velocity of a wave of *some* frequency and generates (surface) waves (except for very small velocities or wavelengths for which surface tension becomes important).[13]

Now we can get a feeling about the design of nonradiating sources; replacing a point by a finite source distribution brings about the possibility of destructive interference for particular values of the wave vector $\mathbf{k}$.

In the case of uniform motion, one must ensure destructive interference in the direction $\theta_0 = \cos^{-1}(c'/v)$ for each angular frequency $\omega$ satisfying $c' = c/n(\omega) < v$.

For a source oscillating at the frequency $\omega$, one must ensure destructive interference for $k = \omega/c'$.

The common experience about antenna and diffraction problems suggests that the design of such sources is not a trivial game. Usual antennas (in vacuum or in a space-time invariant dielectric with refraction index $n > 0$) have radiation patterns such that they do not radiate in *certain* directions, but they generally radiate in *some* direction. The same is true for the diffraction pattern of a distribution of apertures in a diffracting screen.

## III. CONDITION FOR ABSENCE OF RADIATION

### A. Source

The space-time source's distribution is defined by the current $\mathbf{J}(\mathbf{r},t)$, i.e., in Fourier space

$$\mathbf{J}(\mathbf{k},\omega) = \int d^3r\, dt\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\mathbf{J}(\mathbf{r},t)$$

and the charge distribution $\rho(\mathbf{r},t)$ satisfies (from conservation of charge)

$$\omega\rho(\mathbf{k},\omega) = \mathbf{k}\cdot\mathbf{J}(\mathbf{k},\omega).$$

We restrict this section to the following cases:

The source occupies a finite volume (included in a sphere of radius $R$) at any given time.

The source either exists during a finite time or is stationary,[14] i.e., either

(1) $\mathbf{J}(\mathbf{r},t) = 0$ for $|t| > T$

or

(2) $\mathbf{J}(\mathbf{r},t)$ is stationary (in particular, it can be periodic).

Consequently, in case (1), $\mathbf{J}(\mathbf{k},\omega)$ is an analytic function of $\omega$; in case (2) it can only be defined as a generalized function of $\omega$.

### B. Medium

Let the ambient medium be space-time invariant, lossless, and without spatial dispersion. The dielectric permittivity is $\epsilon = \epsilon(\omega) + io\,\text{sgn}(\omega)$, where the infinitesimal imaginary part stands for either infinitesimal losses or a causality condition, equivalent to choosing the so-called retarded solution of Maxwell's equations (except in some special media where this point must be more carefully settled[15]); let $n = \epsilon(\omega)^{1/2}$.

### C. Radiation

From Maxwell's equations, the source's electric field is (in Fourier space)

$$\mathbf{E}(\mathbf{k},\omega) = \frac{-i}{\epsilon_0}\left(\frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{J})/k^2}{\omega\epsilon} + \omega\frac{\mathbf{J} - \mathbf{k}(\mathbf{k}\cdot\mathbf{J})/k^2}{\omega^2\epsilon - k^2c^2}\right). \quad (1)$$

The first term is the longitudinal field ($\mathbf{E}\|\mathbf{k}$); in the special case of a static charge distribution, it gives the Coulomb field. The second term, which represents the transverse field, has a pole whose real part is $\text{Re}(k) = n\omega/c$, which causes the electromagnetic radiation.

The instantaneous source's power loss is

$$P(t) = -\int d^3r\, \mathbf{E}(\mathbf{r},t)\cdot\mathbf{J}(\mathbf{r},t). \quad (2)$$

It is important to note that this is not necessarily the same thing as the power escaping at infinity, i.e., the radiation, which is found by integrating the radial component of the Poynting vector over the surface of an infinite sphere. The difference is the time variation of the field energy in the medium, which may be different from zero if either $\epsilon$ has a finite imaginary part (loss or amplification in the medium), has a zero in the spectral range of the source (resonance), or if the problem is nonstationary. Confusion between these two quantities has led to some controversies and paradoxes.[16,17]

We will discard the first term in Eq. (1) because it does not contribute to the radiation (although in the cases noted above it can contribute to the source's power loss.[17]

Now, depending on the nature of the source, either the total energy loss (during an infinite time) is finite [case (1) in Sec. III A], or the mean power is finite [case (2)] while the total energy is infinite. So we calculate in case (1)

$$W = \int_{-\infty}^{\infty} dt\, P(t) \quad (3)$$

and in case (2),

$$\langle P\rangle = \lim_{\tau\to\infty}\frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} dt\, P(t). \quad (4)$$

Using Parseval's relation, we get from (2)

$$\int_{-\infty}^{+\infty} dt\, P(t) = \frac{-1}{(2\pi)^4}\int d\omega\, d^3k\, \mathbf{E}(\mathbf{k},\omega)\cdot\mathbf{J}^*(\mathbf{k},\omega). \quad (5)$$

Now, inserting the transverse part of (1) in (5), and using (3) [case (1)] or (4) [case (2)] gives the total energy [case (1)] or the mean power loss [case (2)] in *radiation* because in the cases considered here the term involving the energy in the medium disappears in the time integration. One gets

$$\int_{-\infty}^{+\infty} dt\, P(t) = \frac{-1}{(2\pi)^4\epsilon_0}\,\text{Im}\left(\int d\omega\, d^3k \times\omega\frac{|\mathbf{J}(\mathbf{k},\omega)|^2 - |\mathbf{k}\cdot\mathbf{J}(\mathbf{k},\omega)|^2/k^2}{\omega^2[n^2 + io\,\text{sgn}(\omega)] - k^2c^2}\right),$$

where Im denotes the imaginary part.

Since the numerator is real, the only contribution to the integral stems from the poles $k = |\omega|n/c + io\,\text{sgn}(\omega)$, and we obtain, by using Plemelj's relations

$$\int_{-\infty}^{+\infty} dt\, P(t) = \frac{1}{2^5\pi^3\epsilon_0 c}\int_{-\infty}^{+\infty}\frac{d\omega}{n} \times\int d\Omega(k^2J^2 - |\mathbf{k}\cdot\mathbf{J}|^2)_{k=n|\omega|/c}, \quad (6)$$

where the second integral ($d\Omega = \sin\theta\, d\theta\, d\phi$) is over the direction of $\mathbf{k}$.

Note that, in case (2), $|J(\mathbf{k},\omega)|^2$ is not strictly defined, even in the sense of generalized functions, since it involves squares of $\delta$ functions; in order to avoid treating this case separately, we will then use the usual physicist's loose trick (see, for instance, Ref. 18) of replacing the undefined quantity $[\delta(\omega)]^2$ by

$$(1/2\pi)\lim_{\tau\to\infty}[\tau\,\delta(\omega)] \quad (7)$$

and view the integral over $t$ in (6) as

$$\lim_{\tau\to\infty}\int_{-\tau/2}^{+\tau/2}$$

for using it in Eq. (4).

### D. No radiation

Since $(k^2J^2 - |\mathbf{k}\cdot\mathbf{J}|^2)_{k=n|\omega|/c} \geqslant 0$, the necessary and sufficient condition for the integral (6) to be zero, and thus [since $P(t)\geqslant 0$] to have $P(t) = 0$ for any $t$ is

$$kJ(\mathbf{k},\omega) = 0 \quad\text{or}\quad \mathbf{J}(\mathbf{k},\omega)\|\mathbf{k}, \qquad \text{for}\quad k = n|\omega|/c, \quad (8)$$

which is a formal generalization of the Goedecke[10] condition.

What does this mean in practice?

If $kJ(\mathbf{k},\omega) = 0$ for $k = n|\omega|/c$, then the source has no Fourier component for electromagnetic waves propagating in the medium, and it is therefore not surprising that it does not radiate.

It is instructive to compare this to classical scalar diffraction problems. In the simplest case, one puts a harmonic (angular frequency $\omega$) point source at infinite distance behind an opaque plane screen with holes. The field's distribution at a given distance on the other side of the screen is the inverse (two-dimensional) Fourier transform of the product of the Fourier transform of the field distribution on the screen by a phase factor. Now, for the field to be zero at any distance, the latter Fourier transform must be identically zero, which requires blocking up all the holes.

If, on the other hand, $\mathbf{J}(\mathbf{k},\omega) \| \mathbf{k}$, then $\nabla\times\mathbf{J} = 0$, and the absence of radiation is linked to the fact that the field is purely longitudinal. (A trivial example of such a source is a spherically symmetrical charge with a purely radial motion.)

## IV. DESIGNING NONRADIATING SOURCES

### A. Uniform motion

Let a charge distribution $\rho(\mathbf{r})$ be moving at uniform velocity $\mathbf{v}$, whence the current

$$\mathbf{J}(\mathbf{r},t) = \mathbf{v}\rho(\mathbf{r} - \mathbf{v}t),$$

$$\mathbf{J}(\mathbf{k},\omega) = \mathbf{v}2\pi\rho(\mathbf{k})\delta(\omega - \mathbf{k}\cdot\mathbf{v}). \quad (9)$$

The condition $kJ(\mathbf{k},\omega) = 0$ for $k = n|\omega|/c$ reads $k\rho(\mathbf{k}) = 0$ for any $\mathbf{k}$ satisfying $k = n|\mathbf{k}\cdot\mathbf{v}|/c$ or, equivalently [since $\rho(-\mathbf{k}) = \rho^*(\mathbf{k})$], $k\rho(\mathbf{k}) = 0$ for any $\mathbf{k}$ making an angle $\theta_0 = \cos^{-1}(c/nv)$ with $\mathbf{v}$ (whenever $k = n|\omega|/c$ is such that $c/nv < 1$).

Can one design a charge distribution satisfying this condition?

Let us first neglect the variation of $n$ with the frequency $\omega$. In this case, a nonradiating charge must satisfy $k\rho(\mathbf{k}) = 0$ for any $\mathbf{k}$ lying on the surface of a cone of vertex angle $\theta_0$ with respect to $\mathbf{v}$.

Then a trivial solution is a uniform charge distribution on the surface of a plane being nowhere perpendicular to the surface of the cone. In effect, let $\mathbf{v}$ be parallel to the $ox$ axis, and take a source $\rho(\mathbf{r}) = q\,\delta(x - y\tan\alpha)$, thus

$$\rho(\mathbf{k}) = q4\pi^2\,\delta(k_z)\delta(k_y + k_x\tan\alpha)$$

(meaning that the source's Fourier components $\mathbf{k}$ are perpendicular to the plane). If $\alpha \neq \theta_0 + p\pi$ ($p$ integer), then the nonradiating condition is satisfied. But this distribution involves charges at infinite distance, contrary to what we have assumed in Sec. III A.

Can we design nonradiating distributions $\rho(\mathbf{r})$ localized at finite distance $r \leqslant R$? Since spherically symmetric charges will not do, the simplest step is to assume cylindrical symmetry around $\mathbf{v}$. Let us take

$$\rho(\mathbf{r}) = F(r)P_l(\cos\theta'), \quad (10)$$

with $\theta'$ being the angle between $\mathbf{r}$ and $\mathbf{v}$, $P_l(\mu)$ being a Legendre polynomial of order $l$,[19] and $F(r)$ being a function equal to zero for $r > R$. Now, we use the fact that $e^{i\mathbf{k}\cdot\mathbf{r}}$ can be expanded as a sum of products of the form

$$G_{lm}(kr)Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi'),$$

where $(\theta,\phi)$ are polar angles of $\mathbf{k}$, $(\theta',\phi')$ are polar angles of $\mathbf{r}$, with respect to the $ox(\|\mathbf{v})$ axis, and $Y_l^m$ are spherical harmonics. Since the latter functions are orthonormal, and $P_l \propto Y_l^0$, we deduce that

$$\rho(\mathbf{k}) \propto P_l(\cos\theta).$$

As a consequence, if the Legendre polynomial satisfies $P_l(\cos\theta_0) = 0$, then the charge distribution (10) will not radiate by Čerenkov emission when moving at velocity $\mathbf{v}$ in the medium of index $n$. For example, for $c/nv = 3^{-1/2}$, we choose $\rho(\mathbf{r}) = F(r)P_2(\cos\theta')$ since $P_2(\mu) = (3\mu^2 - 1)/2$; for $c/nv = (3/5)^{1/2}$, we choose $P_3$, since $P_3(\mu) = (5\mu^3 - 3\mu)/2$, and so on.

This can be trivially generalized to nonsymmetrical charges: A charge of the form

$$\rho(\mathbf{r}) = F(r)Y_l^m(\theta',\phi') + (-1)^m F(r)^* Y_l^{-m}(\theta',\phi'), \quad (11)$$

where $\cos\theta_0$, is a zero of the associated Legendre function $P_l^m(\cos\theta')$ does not radiate by Čerenkov emission. Note in passing that the possibility of finding a $P_l^m$ for any value of $c/nv$ requires that the set of all zeros of those functions be dense over the interval $[-1, +1]$: We have not explored this question.

Now, what happens when the dispersion is taken into account? In this case, the angle $\theta_0 = \cos^{-1}(c/nv)$ becomes a function of $\omega$ and thus of $k$, which tends to $\pi/2$ when $k \to \infty$: As a consequence, a source such as (11) will radiate in the part of the frequency spectrum where $c/nv$ is no longer a zero of $P_l^m$.

Note that Eq. (6) can be used to give readily the Cerenkov radiation of various charge distributions $\rho(\mathbf{r})$. Inserting (9) into (6), we obtain by using (4) and (7)

$$\langle P\rangle = \frac{1}{2^3\pi^2\epsilon_0 c}\int_0^\infty \frac{d\omega}{n}\int d\Omega \times [(k^2v^2 - |\mathbf{k}\cdot\mathbf{v}|^2)\delta(\omega - \mathbf{k}\cdot\mathbf{v})|\rho(\mathbf{k})|^2]_{k = n\omega/c}.$$

Integrating over the angle $\theta$ between $\mathbf{k}$ and $\mathbf{v}$, with $d\Omega = \sin\theta\,d\theta\,d\phi$ and $\rho(\mathbf{k}) = \rho(k,\theta,\phi)$, we obtain

$$\langle P\rangle = \frac{v}{2^3\pi^2\epsilon_0 c^2}\int_{n > c/v} d\omega\omega\left(1 - \frac{c^2}{n^2v^2}\right) \times\int_0^{2\pi} d\phi\left|\rho\left(\frac{n\omega}{c},\theta_0,\phi\right)\right|^2, \quad (12)$$

where $\theta_0 = \cos^{-1}(c/nv)$ and the integral is over the positive frequencies satisfying $n(\omega) > c/v$.

In the special case of a cylindrically symmetric (with respect to $\mathbf{v}$) charge distribution $\rho(k,\theta)$, (12) takes the simple form

$$\langle P\rangle = \frac{v}{4\pi\epsilon_0 c^2}\int_{n > c/v} d\omega\,\omega\left(1 - \frac{c^2}{n^2v^2}\right)\left|\rho\left(\frac{n\omega}{c},\theta_0\right)\right|^2. \quad (13)$$

For a point charge $q[\rho(\mathbf{k}) = q]$, (13) reduces to the well-known Frank–Tamm result. Let us now take a charge $q$ distributed over a spherical shell of radius $R$, i.e.,

$$\rho(\mathbf{r}) = q\,\delta(r - R)/4\pi R^2,$$

$$\rho(\mathbf{k}) = q\sin(kR)/kR. \quad (14)$$

Inserting (14) into (13) shows that, as expected, the finite source's size $R$ lowers the contribution of frequencies $\omega > c/nR$. But this is not sufficient to make the integral (13) converge for $\omega \to \infty$, so that it is still necessary to take the dispersion into account in order to ensure convergence at large $\omega$.

### B. Small oscillations

We take a charge distribution $\rho(\mathbf{r})$ that oscillates at the (angular) frequency $\omega$, namely, $\mathbf{J}(\mathbf{r},t) = (\partial\mathbf{d}/\partial t)\rho[\mathbf{r} - \mathbf{d}(t)]$, with $\mathbf{d}(t) = \mathbf{d}\sin\omega_0 t$. We assume $n\omega_0 d/$

$c \ll 1$ (the so-called dipole approximation), so that the Fourier transform simplifies to

$$\mathbf{J}(\mathbf{k},\omega) \simeq i\rho(\mathbf{k})\mathbf{a}(\omega)/\omega, \quad (15)$$

where $\mathbf{a}(\omega)$ is the Fourier transform of the acceleration $\mathbf{a}(t) = \partial^2\mathbf{d}/\partial t^2$; thus

$$\mathbf{J}(\mathbf{k},\omega) \simeq \pi\omega_0\mathbf{d}\,\rho(\mathbf{k})[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]. \quad (16)$$

Inserting (16) into (8) yields the nonradiating condition

$$\rho(\mathbf{k}) = 0 \quad \text{for} \quad k = n\omega_0/c. \quad (17)$$

In the particular case of spherical symmetry, (17) takes the form

$$\int_0^\infty dr\, r \sin\left(\frac{n\omega_0 r}{c}\right)\rho(r) = 0, \quad (18)$$

which is satisfied by

$$\rho(r) \propto \cos(pn\omega_0 r/c)/r, \quad \text{for } r \leqslant R, \quad n\omega_0 R/c = 2\pi l,$$

$$\rho(r) \propto \sin(pn\omega_0 r/c)/r, \quad \text{for } r \leqslant R, \quad n\omega_0 R/c = 2\pi l, \quad p \neq 1$$

($p$ integer $\geqslant 0$, $l$ integer $> 0$), or any linear combination of such functions, and also by

$$\rho(r) \propto \delta(r-R), \quad n\omega_0 R/c = \pi p$$

($p$ integer $> 0$). Note that these distributions have a characteristic scale on the order of the wavelength or a multiple of it.

These results are not restricted to small oscillations: They are easily generalized to any *periodic* motion of period $2\pi/\omega_0$.[10] In this case, the source also exhibits higher harmonic frequencies so that one has to replace $\omega_0$ by every integer multiple of $\omega_0$ in the nonradiating conditions (17) and (18).

Note in passing that inserting (16) into (6) [using (4) and (7)] gives readily the mean power radiated (in the dipole approximation) by an oscillating charge distribution $\rho(\mathbf{k})$ as

$$\langle P\rangle = \frac{\omega_0^2}{2^5\pi^2\epsilon_0 cn}\int d\Omega[(k^2d^2 - |\mathbf{k}\cdot\mathbf{d}|^2)|\rho(\mathbf{k})|^2]_{k=n\omega_0/c}, \quad (19)$$

which gives in the particular case of spherical symmetry

$$\langle P\rangle = (n\omega_0^4 d^2/12\pi\epsilon_0 c^3)|\rho(n\omega_0/c)|^2.$$

For a point charge $[\rho(\mathbf{k}) = q]$, this gives the usual dipolar radiation formula. For a charge of finite size $R$ [use (14), for instance], the power radiated is much smaller than for a point source if $R \gg c/n\omega$.

### C. Nonlocalized sources

The sources considered up to now were localized in a finite volume. One can also easily design nonradiating distributions involving infinite planes or cylinders. Several such harmonic sources have been discussed recently[20] by using retarded potentials in empty space; it is, however, simpler to use the Fourier transforms directly.

The space Fourier transforms now involve generalized functions so that one must use a trick analogous to (7) when integrating along the spatial coordinate(s) where the distribution extends to infinite distance; this ensures the integrations over $\mathbf{k}$ to be properly defined in order to generalize the nonradiation condition.

Consider a harmonic ($\omega_0$) current distributed uniformly on a cylindrically symmetric wire,

$$\mathbf{J}(\mathbf{r},t) = \mathbf{e}_z\, j f(u)\sin\omega_0 t,$$

where $u$ denotes the distance to the $oz$ axis. Thus

$$\mathbf{J}(\mathbf{k},\omega) = -i\mathbf{e}_z 4\pi^3 j\,\delta(k_z)[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)] \times \int_0^\infty du\, uf(u)J_0\left(\frac{n\omega_0 u}{c}\right) \quad (20)$$

so that the nonradiating condition (8) becomes

$$\int_0^\infty du\, uf(u)J_0\left(\frac{n\omega_0 u}{c}\right) = 0,$$

which is satisfied by $f(u) = \delta(u-R)$, where $n\omega_0 R/c$ is a zero of $J_0$ and $f(u) = J_0(\alpha u/R)(u \leqslant R)$, where $n\omega_0 R/c$ and $\alpha$ are distinct zeros of $J_0$.

Now, what about plane distributions? We have already seen that a charge uniformly distributed on an infinite plane does not radiate by Čerenkov emission if the plane is nowhere perpendicular to the surface of the Čerenkov cone.

Now, consider a harmonic current uniformly distributed on a plane $xoy$: It always radiates. In effect, $J(\mathbf{k},\omega) \propto \delta(k_x)\delta(k_y)[\ \ ]$ [where the term in brackets is the same as in (20)] so that condition (8) cannot be satisfied. On the other hand, one can build a nonradiating source with several such parallel planes[20] since $J(\mathbf{k},\omega)$ is now multiplied by a function of $k_z$ which can be zero for $k_z = n\omega_0/c$, if the planes have the right separation (i.e., an odd multiple of $\pi c/n\omega_0$).

## V. CONCLUSION

One is accustomed to thinking that a charged body, which is accelerated and/or moves faster than light in a medium, radiates. We have seen that this intuition is not necessarily correct for the idealized problem of a classical rigid source in a nondispersive dielectric.

To ensure this absence of radiation, one must carefully design the source distribution in order that destructive interference occur in every direction. Such special design yielding counterintuitive behavior is reminiscent of the so-called diffraction-free beams whose peculiar structure ensures minimum diffractive spreading in free space[21]; this latter problem is, however, more likely to have practical applications.

A nonradiating source is trivially achieved with a spherically symmetric charge undergoing radial pulsations: The absence of radiation stems from the fact that the field is then purely longitudinal. One can also build a nonradiating oscillating source by ensuring that it has no Fourier components for the wavenumbers propagating in the medium: The size of the source is then of the order of the wavelength in the medium (or a multiple of it). Finally, one can build a nonradiating Čerenkov source by ensuring that it has no Fourier components along the Čerenkov ($\mathbf{k}$) cone.

An alternative method for building "black" sources, which has apparently escaped physicists' attention, had been suggested a long time ago.[22]

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[22] "Our Second Experiment," the Professor announced, ... "is the production of that seldom-seen-but-greatly-to-be-admired phenomenon, Black Light! You have seen White Light, Red Light, Green Light, and so on: but never, till this wonderful day, have any eyes but mine seen *Black Light*! This box ..., is quite full of it. The way I made it was this: I took a lighted candle into a dark cupboard and shut the door. Of course the cupboard was then full of *Yellow* light. Then I took a bottle of black ink, and poured it over the candle: and, to my delight, every atom of the yellow light turned *Black*!" [Lewis Carroll, in *Sylvie and Bruno Concluded* (Vintage, New York, 1976), pp. 712–713].

# Wobbling, toppling, and forces of contact

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Analyses and experiments are described for two familiar systems that upon close inspection reveal some surprises. The wobbling domino provides a simple model for some important features in the mechanics of walking. It also makes a sensitive level. The toppling pencil calls for careful treatment of friction.

## LIST OF SYMBOLS

(Defining equations are noted in parentheses)

| | |
|---|---|
| $b$ | base width of wobbling object (Fig. 2) |
| $E$ | energy (26) |
| $F_x$ | vertical force (25) |
| $F_y$ | horizontal force (25) |
| $g$ | gravitational acceleration |
| $h$ | height to mass center of wobbling object (Fig. 2) |
| $k$ | step index |
| $L$ | left edge (Fig. 2) |
| $l$ | length from edge to mass center (Figs. 2 and 7) |
| $m$ | object mass |
| $R$ | right edge (Fig. 2) |
| $r_{gyr}$ | radius of gyration about the mass center |
| $T$ | total duration of wobbling (14) |
| $\mathbf{V}_{CM}$ | velocity of the mass center (29) |
| $\mathbf{V}_F$ | velocity of the foot (30) |
| $x$ | vertical coordinate (Fig. 7) |
| $y$ | horizontal coordinate (Fig. 7) |
| $\alpha_b$ | half-angle between legs (Fig. 2) |
| $\alpha_L$ | contact angle on left edge (Fig. 2) |
| $\alpha_R$ | contact angle on right edge (Fig. 2) |
| $\alpha_k$ | contact angle for the $k$th step (8) |
| $\gamma$ | slope |
| $\eta$ | coefficient of restitution (6) |
| $\theta$ | leg angle (Figs. 2 and 7) |
| $\theta_k$ | peak leg angle during the $k$th step |
| $\mu_d$ | coefficient of sliding friction (37) |
| $\mu_s$ | coefficient of static friction (37) |
| $\sigma$ | time-scale parameter (2) |
| $\tau$ | dimensionless time, $t\sqrt{g/l}$ |
| $\Omega$ | dimensionless speed, $d\theta/d\tau$ |
| $\Omega_0$ | initial speed |
| $\Omega^-$ | speed just before support transfer |
| $\Omega^+$ | speed just after support transfer |

  