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Chaotic motions of Prometheus and Pandora

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Abstract

Recent HST images of the saturnian satellites Prometheus and Pandora show that their longitudes deviate from predictions of ephemerides based on *Voyager* images. Currently Prometheus is lagging and Pandora leading these predictions by somewhat more than 20°. We show that these discrepancies are fully accounted for by gravitational interactions between the two satellites. These peak every 24.8 days at conjunctions and excite chaotic perturbations. The Lyapunov exponent for the Prometheus–Pandora system is of order 0.3 year^{-1} for satellite masses based on a nominal density of 0.63 g cm^{-3}. Interactions are strongest when the orbits come closest together. This happens at intervals of 6.2 years when their apses are antialigned. In this context, we note the sudden changes of opposite signs in the mean motions of Prometheus and Pandora at the end of 2000 occurred around the time their apsidal lines were antialigned.

Keywords: Satellites of Saturn; Orbits; Chaos

1. Introduction

Orbits for Pandora and Prometheus in the form of precessing ellipses of fixed shape were fitted to *Voyager* data by Synnott et al. (1981, 1983) and Jacobson (personal communication). Mean motions were determined from images and precession rates were calculated to be consistent with the gravity field of the saturnian system (Nicholson and Porco, 1988; Campbell and Anderson, 1989).

Observations with HST made during the 1995–1996 Sun and Earth ring plane crossings led to the discovery that Prometheus was lagging its predicted longitude based on the *Voyager* ephemeris by about 20° (Bosh and Rivkin, 1996; Nicholson et al., 1996). Subsequently McGhee (2000) found that Pandora was leading the *Voyager* ephemeris prediction by a similar amount. These discrepancies have been confirmed by French et al. (1999, 2000, 2001, 2002), Murray et al. (2000), McGhee et al. (2001), and Evans (2001).

These and other researchers looked for a dynamical origin of the longitude discrepancies. Several hypotheses, including perturbations exerted by an undetected coorbital satellite of Prometheus (see French et al., 1998); interactions with clumps in the F ring, or 1- to 5-km objects in the F ring, or the F ring itself (Showalter et al., 1999a, b); long-term resonance dynamics (Dones et al., 1999); and chaos (Dones et al., 2001), were investigated. However, none of these attempts provide a clear resolution of this puzzle. That is the goal of our paper.

We focus on direct interactions between Pandora and Prometheus because their longitude discrepancies have comparable magnitudes and opposite signs (French et al., 2002). This suggests that the satellites are exchanging angular momentum and energy and that their orbits are chaotic. Results from orbit integrations presented in Section 2 of the paper confirm this suspicion. We review prior suggestions that the motions of Prometheus and/or Pandora might be chaotic in Section 3. This section also reproduces evidence from French et al. (2002) that supports our finding that sudden changes in mean motion tend to occur around times when the satellites' apses are antialigned.

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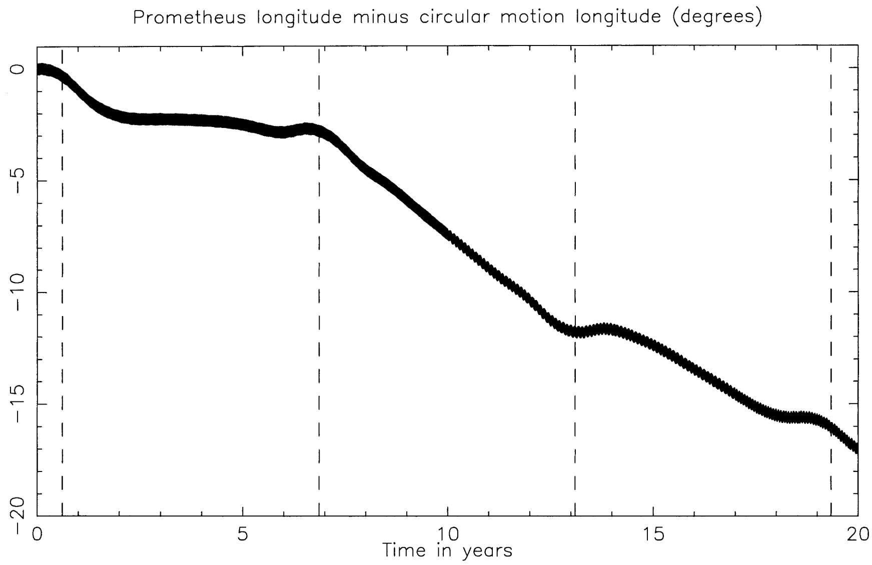

Fig. 1. Prometheus longitude from numerical integration as a function of time. A drift based on the initial mean motion is subtracted from the longitude. Units are degrees and years. The dashed lines indicate the time of periapsis antialignment.

2. Confirmation of chaos

2.1. Calculational method

To observational accuracy, the orbits of Prometheus and Pandora lie in Saturn's equatorial plane. Working in a planet-centered coordinate system and adopting conventional notation, the vector equation for equatorial motion reads

$$\frac{d^2 r_i}{dt^2} = -\frac{GMr_i}{r_i^3}\left[\left(1+\frac{m_i}{M}\right)\frac{3}{2}J_2\left(\frac{R}{r_i}\right)^2 - \frac{15}{8}J_4\left(\frac{R}{r_i}\right)^4 + \frac{35}{16}J_6\left(\frac{R}{r_i}\right)^6\right] - Gm_j\left(\frac{r_i - r_j}{|r_i - r_j^3|} + \frac{r_j}{r_j^3}\right), \quad (1)$$

where i and j ($i \neq j$) assume values 1 and 2.

Equations (1) admit energy and angular momentum integrals given by

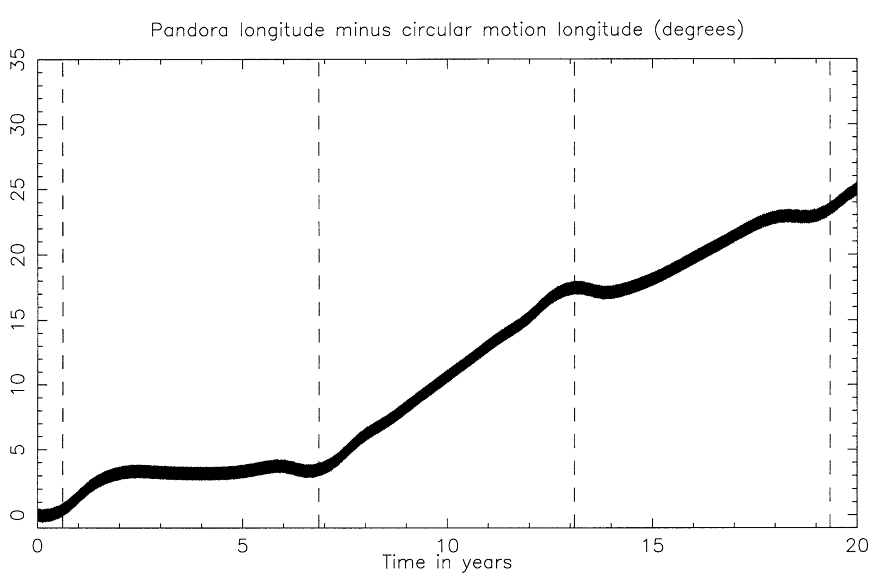

Fig. 2. Pandora longitude from numerical integration as a function of time. A rate based on the initial mean motion is subtracted from the longitude. Units are degrees and years. The dashed lines indicate the time of periapsis antialignment.

Table 1
Initial values for the eccentricities, mean motions, mean longitudes, and masses scaled to Saturn's mass for Prometheus and Pandora

Satellite	e_0	n_0 (rd/s)	λ_0 (°)	m/M
Prometheus	2.29×10^{-3}	1.1864×10^{-4}	189	5.80×10^{-10}
Pandora	4.37×10^{-3}	1.1571×10^{-4}	82	3.43×10^{-10}

$$E = \frac{1}{2}\left[m_1\left|\dot{r}_1\right|^2 + m_2\left|\dot{r}_2\right|^2 - \frac{|m_1\dot{r}_1 + m_2\dot{r}_2|^2}{M + m_1 + m_2}\right] - \frac{Gm_1m_2}{|r_1 - r_2|} - \frac{GMm_1}{r_1}\left[1 - \frac{1}{2}J_2\left(\frac{R}{r_1}\right)^2 + \frac{3}{8}J_4 \times \left(\frac{R}{r_1}\right)^4 - \frac{5}{16}J_6\left(\frac{R}{r_1}\right)^6\right] - \frac{GMm_2}{r_2} \times \left[1 - \frac{1}{2}J_2\left(\frac{R}{r_2}\right)^2 - \frac{3}{8}J_4\left(\frac{R}{r_2}\right)^4 - \frac{5}{16}J_6\left(\frac{R}{r_2}\right)^6\right], \tag{2}$$

and

$$H = m_1(r_1 \times \dot{r}_1) + m_2(r_2 \times \dot{r}_2) - \frac{(m_1r_1 + m_2r_2) \times (m_1\dot{r}_1 + m_2\dot{r}_2)}{M + m_1 + m_2}. \tag{3}$$

Numerical integrations of the equations of motion are carried out using the algorithm of Bulirsch and Stoer (1980), which offers the luxury of a variable time step. Fractional changes in total energy and angular momentum are of order 10^{-10} for integrations of 10^3 years. For comparison, jumps in these quantities are of order 10^{-6} at each conjunction.

Initial conditions are computed from Jacobson's equinoctial elements (Jacobson, personal communication) and the transformation between cylindrical elements and epicyclic elements derived in Borderies-Rappaport and Longaretti (1994). The same transformation is applied to compute the epicyclic eccentricity and the epicyclic mean longitude at each output step. We also output values of the angular momentum and energy (neglecting the interaction term) for each satellite, and the total angular momentum and energy (including the interaction term).

To compute Lyapunov exponents we integrate the orbits of two shadow bodies whose initial conditions differ slightly from those of Prometheus and Pandora. We reset the state vector of the shadow bodies to reduce the magnitude of their phase space separation from the physical bodies whenever it exceeds a preset tolerance. In practice rescaling is done when the longitudinal separations that dominate the configuration space separations are somewhat less than 10^{-4} rad.

2.2. Results

Variations over 20 years of orbital longitudes for Prometheus and Pandora are displayed in Figs. 1 and 2. Initial values for the satellites' epicyclic eccentricities and mean motions obtained following the prescription described in Section 2.1 are presented in Table 1; this table also

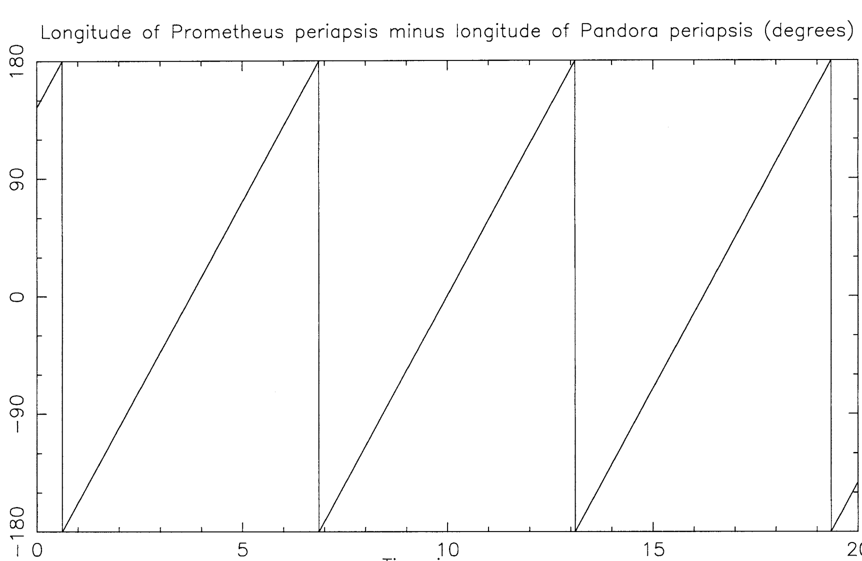

Fig. 3. Difference between the epicyclic apsidal longitudes (in degrees) of Prometheus and Pandora over 20 years.

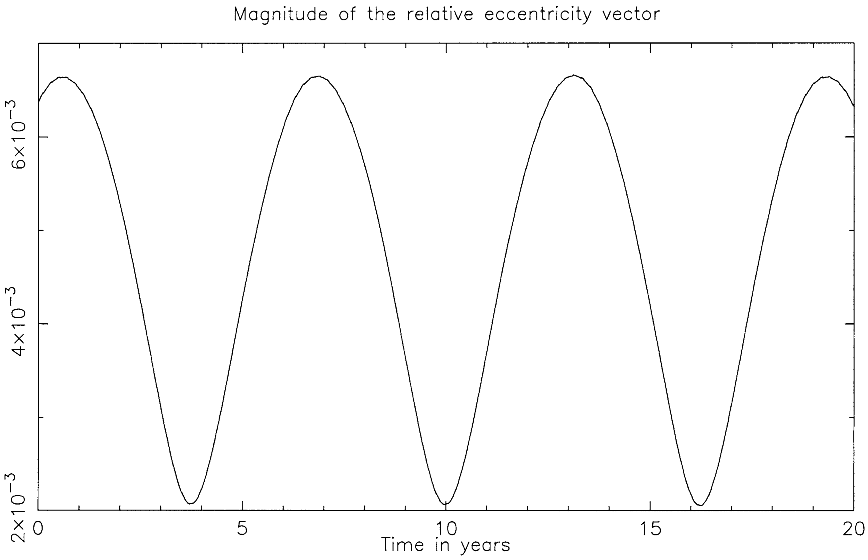

Fig. 4. Magnitude of the Prometheus and Pandora relative eccentricity vector. Note that the peaks correspond to antialigned apses.

contains the ratios of the satellites' masses to Saturn's mass.[1]

The simulation begins with Prometheus and Pandora at their locations on 1981 August 23 04:02:12 UTC, the epoch of Jacobson's ephemerides.[2] To emphasize the chaotic irregularities of the mean motions, we subtract $\lambda_0 + n_0(t - t_0)$ from the longitude of each satellite. These figures reproduce the characteristics of the puzzling longitude discrepancies reported in the papers referenced in Section 1. Line widths are due to epicyclic longitude oscillations that have full amplitudes of $4e$ radians.

Fig. 3 displays the difference between the apsidal angles over 20 years, and Fig. 4 shows the behavior of the magnitude of the relative eccentricity vector. The two orbits

[1] The shapes of Prometheus and Pandora were determined by Thomas (1989) in the form of triaxial ellipsoids. The principal axes were determined with formal uncertainties of 1.5 and 0.6 km, respectively. The densities of both satellites are unknown, so Epimetheus' density of 0.63 g cm^{-3} (Nicholson et al., 1992) was arbitrarily adopted. Plausible density uncertainties of 50% would lead to formal uncertainties of 59 and 54% in the masses of Prometheus and Pandora, respectively.

[2] We denote the time at epoch by t_0.

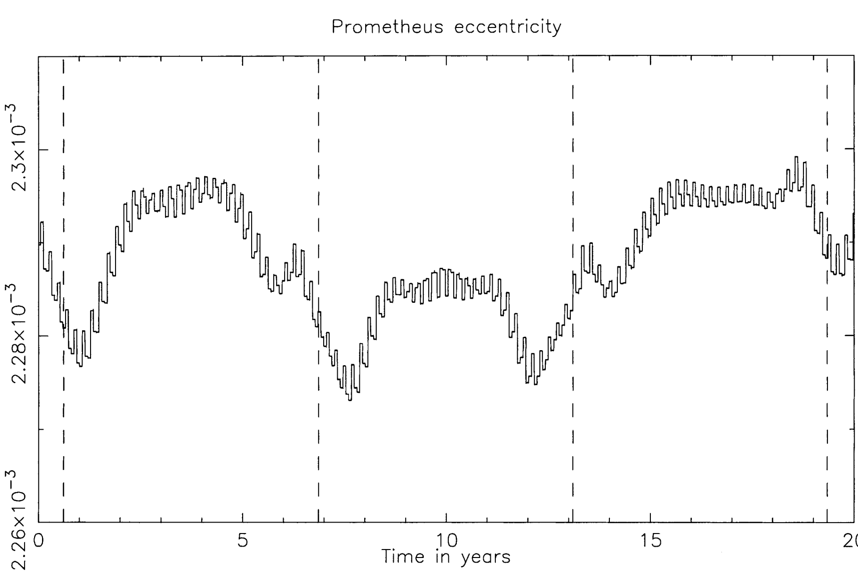

Fig. 5. Prometheus epicyclic eccentricity as a function of time. The epicyclic frequency is computed from the state in rectangular coordinates following Borderies-Rappaport and Longaretti (1994). The dashed lines indicate the time of periapsis antialignment.

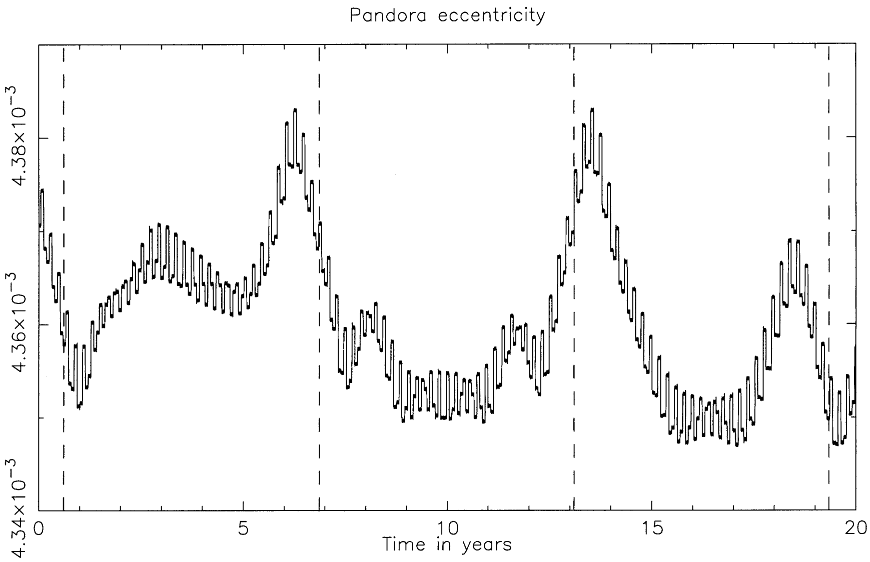

Fig. 6. Pandora epicyclic eccentricity as a function of time. The epicyclic frequency was computed from the state in rectangular coordinates following Borderies-Rappaport and Longaretti (1994). The dashed lines indicate the time of periapsis antialignment.

come closest together and the magnitude of the relative eccentricity peaks when the apsidal lines are antialigned. This occurs at about $t = 0.61$, 6.9, 13.1, and 19.3 years. The times of antialignment are indicated with dashed lines in Figs. 1 and 2. It is apparent that these are the times at which abrupt changes in the satellites' mean motions take place. Note the different magnitudes of the net changes of mean motions that occur around times of antialignment. This is another indication of chaos.

Additional evidence for chaos is found in plots of eccentricity vs time shown in Figs. 5 and 6. Two distinct types of eccentricity variation are apparent. Small jumps occur at conjunctions separated by about 24.8 days. These have magnitudes of order $\mu\,(a/\Delta r)^2 \sim 5 \times 10^{-6}$, where μ is the mass of the perturbing satellite divided by the mass of Saturn, a is the mean orbit radius, and Δr is the radial distance between the satellites at conjunction. As expected, the largest jumps occur when the satellites' apses are near antialignment. Quasiperiodic variations of eccentricity are associated with the relative apsidal precession period of 6.2 years. They arise from secular perturbations that promote the exchange of angular momentum but not of energy.

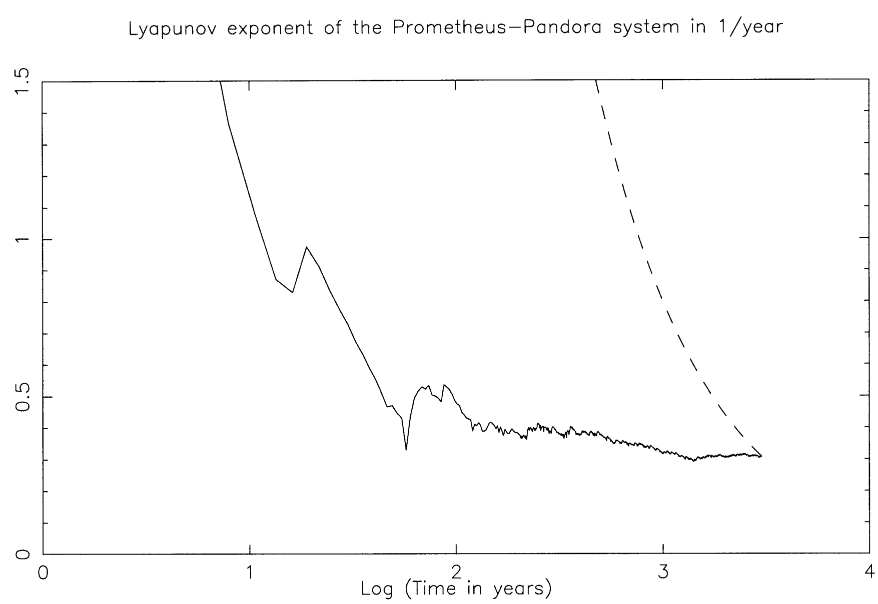

Fig. 7. Lyapunov exponent for the Prometheus–Pandora system over a period of 3×10^3 years (solid line). The dashed line depicts a constant + (log t) /t fitted to the final point of the solid curve. The unit for the Lyapunov exponent is year^{-1}.

Although the secular variations are somewhat larger than the jumps, they are small in comparison to the mean eccentricity. Their small size is a consequence of the dominance of Saturn's oblate gravitational equipotentials in forcing the differential precession; secular terms in the satellites' interaction potential contribute only a small fraction of the differential precession rate. Eccentricity jumps and secular eccentricity variations are not the entire story, nor even the most important part of it. That distinction goes to the lack of periodicity over the differential precession cycle, which is a clear signature of chaos.

To prove that the mean motion variations arise from chaos, we compute the Lyapunov exponent for the Prometheus–Pandora system. Fig. 7 illustrates its behavior over an interval of 3000 years. The figure also includes a dashed line showing a constant plus (log t) $/t$ fit to the final point of the solid curve. This is the behavior that would be expected in the absence of chaos. Evidence for chaos is overwhelming. The Lyapunov exponent is of order 0.3 year^{-1}.

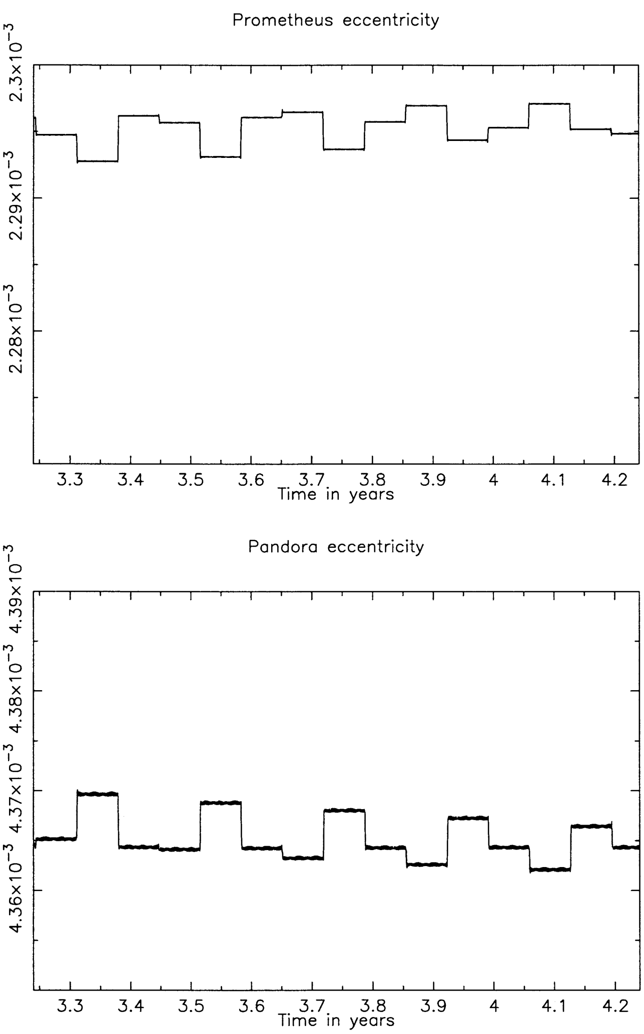

Fig. 8. Prometheus and Pandora epicyclic eccentricities during an interval of one year centered on $t = 3.74$ years when the apses are aligned.

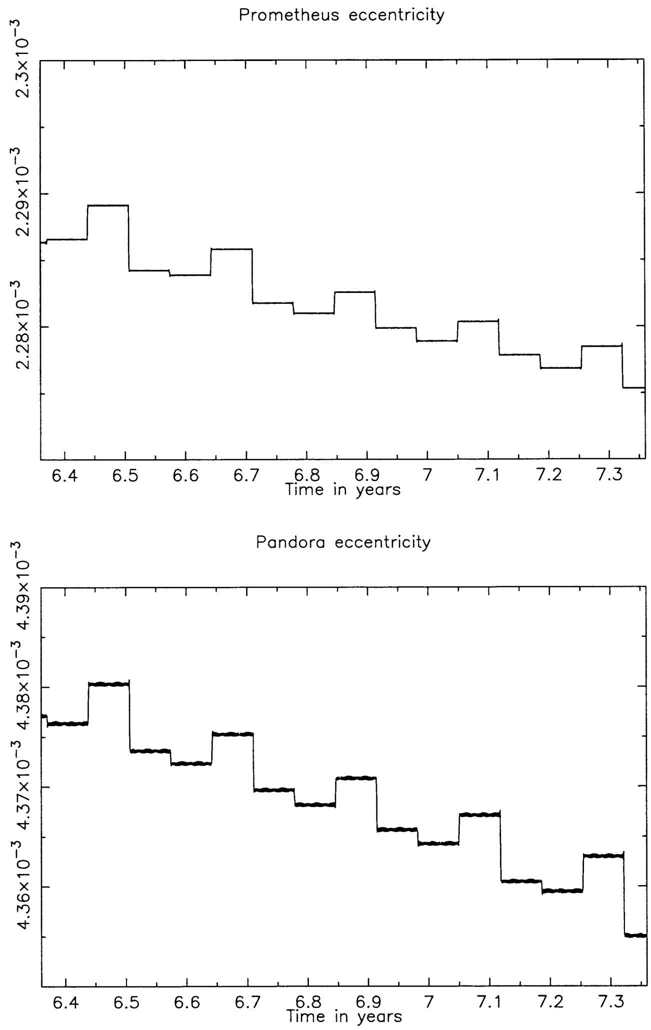

Fig. 9. Prometheus and Pandora epicyclic eccentricities during an interval of one year centered on $t = 6.86$ years when the apses are antialigned.

Figs. 8–11 derived from data spaced by 0.1 d show perturbations near conjunctions in greater detail. Variations of epicyclic eccentricity are depicted in Figs. 8 and 9 while Figs. 10 and 11 provide data on energy and angular momentum accrued during the numerical integration. Each panel covers an interval of one year centered either on $t = 3.74$ years, when the apses are aligned, or on $t = 6.86$ years, when they are antialigned. Perturbations are noticeably larger during the latter than during the former. Fractional jumps of the energy and angular momentum of each satellite at conjunctions are $\sim \mu\Delta e\ (a/\Delta r)^3 \sim 2 \times 10^{-6}$.[3] That our estimates are reasonable can be seen by noting that the energy and angular momenta of Prometheus and Pandora are $\sim 4 \times 10^{32}$ and $\sim 6 \times 10^{36}$ g cm^2 s^{-1}, and that their jumps are $\sim 10^{27}$ and $\sim 10^{31}$ g cm^2 s^{-1}. Spikes seen in the plots of energy and angular momentum arise from the

[3] Δe is the magnitude of the relative eccentricity vector.

strong interactions near conjunctions. Their widths of a few hours are marginally resolved.

3. Discussion

The suggestion that interactions between Prometheus and Pandora make their motions chaotic is not new. It was raised long ago in an article we wrote with Scott Tremaine (Borderies et al., 1984). For us its confirmation is almost like a dream come true.

Recent discussions that address the possibility of chaos can be found in Poulet and Sicardy (2001), Dones et al. (2001), and French et al. (2002). Poulet and Sicardy (2001) investigate the long-term evolution of the system and find intervals of chaos. Dones et al. (2001) suggest that chaos might account for unexplained motions of the satellites but do not identify the specific mechanism responsible for creating it. French et al. (2002) raise the possibility that changes of opposite sign in the mean motions of Prometheus and Pandora may signal the exchange of energy between their orbits. However, they do not simulate the effects of interactions between the shepherds. Instead they present evidence that the co-orbital satellites and Mimas can excite chaotic motions of test particles in a portion of a region of 2×10^3 km width covering the semimajor axes of both shepherds.

We close this paper by displaying evidence in support of our finding that abrupt changes in mean motions tend to occur around times during which the satellite' apses are antialigned. Fig. 12 reproduces, with embellishments, panels from Figs. 3 and 5 of French et al. (2002). It shows that the mean motions of Prometheus and Pandora underwent changes of opposite sign around the end of year 2000, which

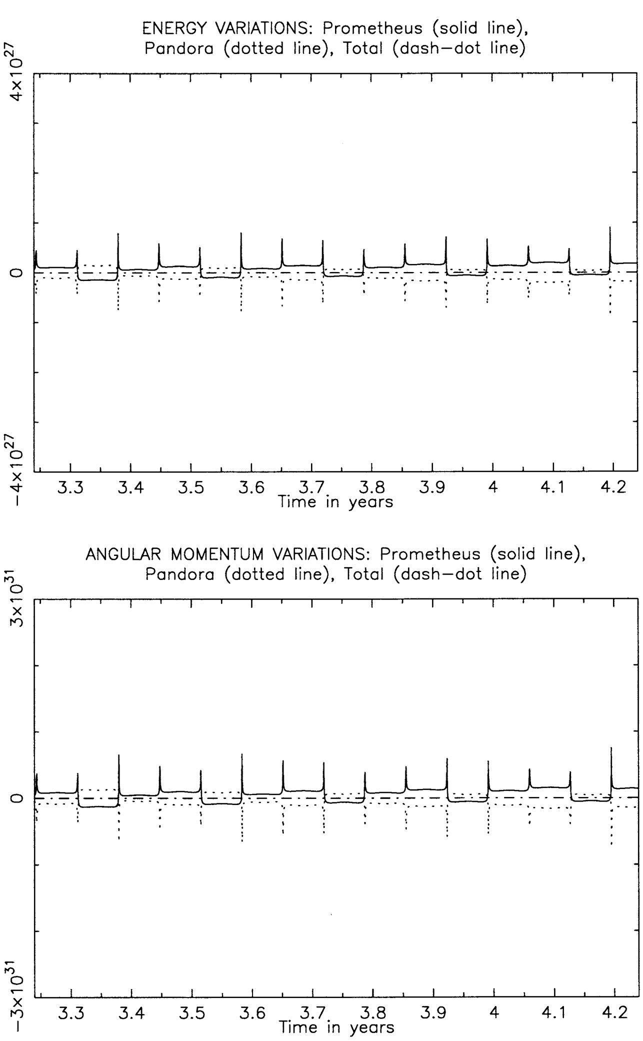

Fig. 10. Prometheus (solid lines) and Pandora (dotted lines) variations in energy (in g cm^2 s^{-2}) and angular momentum (in g cm^2 s^{-1}) during an interval of one year centered on $t = 3.74$ years when the apses are aligned. The dot-dashed lines display differences between current and initial total energies and angular momenta.

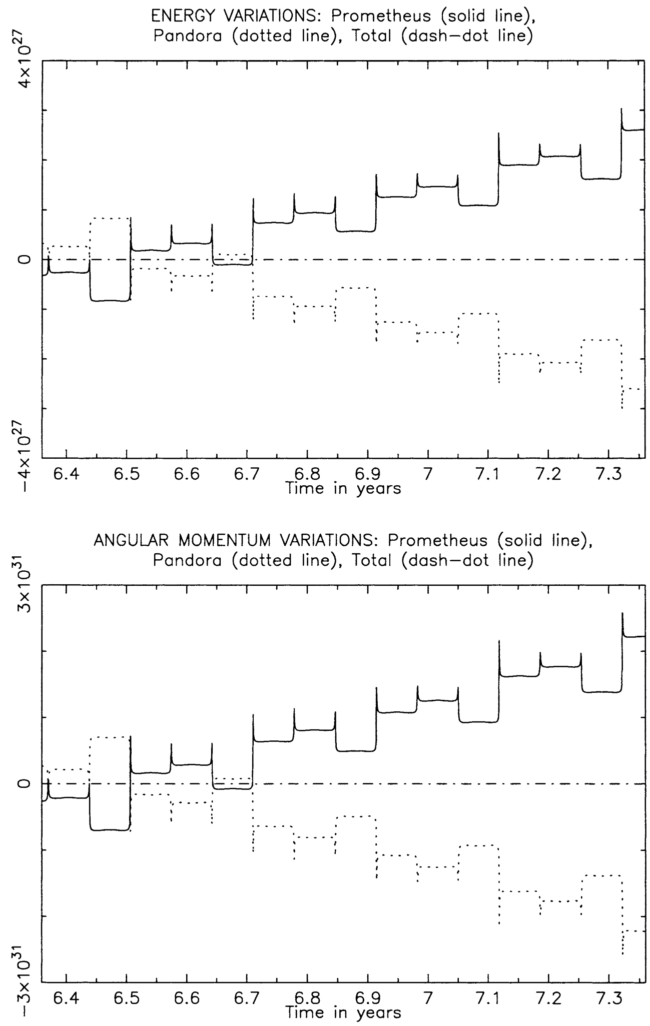

Fig. 11. Prometheus (solid lines) and Pandora (dotted lines) variations in energy (in g cm^2 s^{-2}) and angular momentum (in g cm^2 s^{-1}) during an interval of one year centered on $t = 6.86$ years when the apses are antialigned. The dot-dashed lines display differences between current and initial total energies and angular momenta.

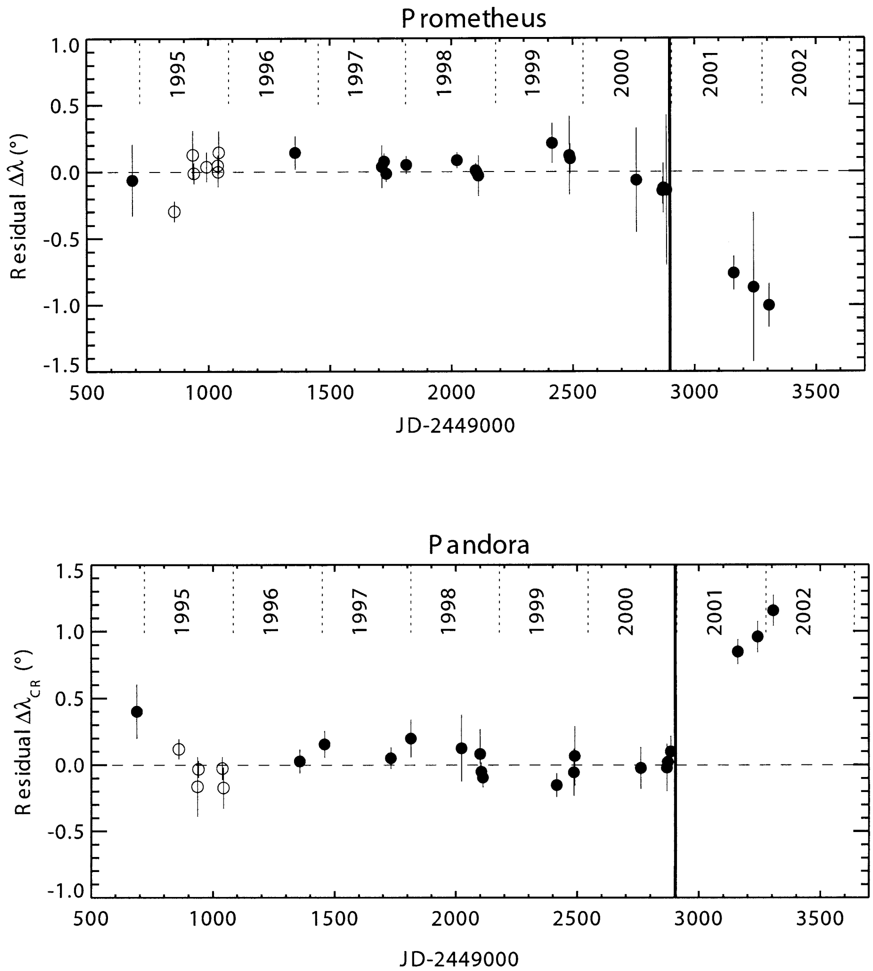

Fig. 12. Evidence for sudden jumps in the mean motions of Prometheus and Pandora at the end of year 2000. Reproduced, with permission, from French et al. (2002). The solid vertical lines mark the time at which the satellites' apses were antialigned.

corresponds to the time at which their apsidal longitudes differed by 180°.

Acknowledgments

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