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# ORBIT - ORBIT RESONANCES IN THE SOLAR SYSTEM: VARIETIES AND SIMILARITIES

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CONTENTS



## 1. INTRODUCTION

Orbit - orbit resonance, the enhancement of mutual interactions of planetary bodies due to repetitive configurations, is a ubiquitous feature of solar system dynamics. Resonances may help explain the structure of the asteroid belt and Saturn's rings (Brouwer, 1963; Franklin and Colombo, 1970), the transport of meteorites to terrestrial planets (Williams, 1973), and perhaps even the original building of planet-sized bodies (Safronov, 1972). The prevalence of resonances permits mass determination for planets and satellites (Duncombe *et al.*, 1973) and places constraints on long-term orbital evolution (reviewed by Peale, 1976). Resonance phenomena manifest themselves in a variety of forms with underlying similarities.

The object of this paper is to bring together descriptions of the various resonance mechanisms in such a manner that the physical processes underlying traditional mathematical presentations are apparent. The goal is an individual perspective rather than an exhaustive review of the subject. My intention is to give the non-specialist a feeling for these physical processes and a taste of the terminology and analytical techniques so that the literature of celestial mechanics will not seem intimidating or opaque. Perhaps my viewpoint will also reinforce the insight of some specialists.

In general, a resonance occurs when the periodic behavior of a dynamical system is matched by some periodic driving force. If a system of planets or satellites has a configurational periodicity, the mutual perturbations will have the same period, thus enhancing the perturbations and yielding an orbit - orbit resonance. A periodicity occurs if "commensurabilities" (small-integer ratios) exist among the orbital periods. The resonance is said to be stable (or "locked") if the enhanced mutual perturbations maintain the commensurability against disruptive influences.

## 2. TITAN - HYPERION

In order to see how such a mechanism can work, consider the following simple model based on the resonance between Saturn's satellites Titan and Hyperion (Fig. 1). The inner satellite of the pair, Titan, has a circular orbit coplanar with that of the outer one, Hyperion, whose orbit has significant eccentricity ($e \simeq 0.1$). The satellites' orbital periods are near a ratio of 3/4. Between conjunctions of the satellites (defined as a configuration with both satellites at the same planetocentric longitude), Titan makes four complete revolutions about Saturn, while Hyperion makes three. This commensurability implies that the longitude of conjunction varies very slowly. (Further details are given by Woltjer, 1928; Goldreich, 1965; and Greenberg, 1973).

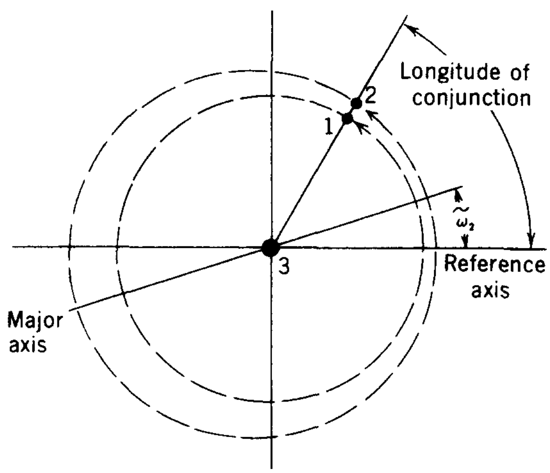


Fig. 1. A simplified model of the orbits of Titan (1) and Hyperion (2). The two satellites are shown in conjunction relative to Saturn (3). The dashed lines depict the orbits and $\tilde{\omega}_2$ represents the outer satellite's longitude of pericenter. The orbit of the inner satellite is circular.

Hyperion's mass is negligible, so its effect on Titan can be ignored. The effect of Titan on Hyperion needs only be considered near conjunction where the satellites are close to one another and the attraction is relatively strong. Suppose conjunction occurs after Hyperion's pericenter and before apocenter, as in Fig. 1. At conjunction Titan exerts a force on Hyperion which is directed radially in toward Saturn, while Hyperion is moving outward as it moves from pericenter to apocenter. Thus energy is removed from Hyperion's motion. Moreover, because the two orbits are diverging at this conjunction, the satellites would actually be closest to one another shortly before conjunction. Titan, having a greater angular velocity, would be behind Hyperion at closest approach and would therefore remove energy from Hyperion's orbit. The loss of energy shrinks Hyperion's orbit and its period. Although this effect is small at each conjunction, it is enhanced by the repetition of this configuration. As Hyperion's period decreases, the ratio of the orbital periods increases above 3/4 so that subsequent conjunctions occur closer to Hyperion's apocenter. Similarly, if conjunction occurs after apocenter, it is driven back towards apocenter. The gravitational interaction tends to maintain conjunction at a certain longitude, i.e. it tends to maintain the commensurability.

The occurrence of conjunction at the longitude of Hyperion's apocenter is a stable configuration. The behavior is closely analogous to that of a pendulum. A pendulum can oscillate about the stable equilibrium position, or, given enough kinetic energy at that position, it can circulate through 360°. Likewise, in the orbital resonance model conjunction can oscillate (or "librate") about Hyperion's apocenter, or, if at the stable configuration the ratio of orbital periods is far enough from commensurability, conjunction can circulate through 360°. The observed Titan - Hyperion case is, in fact, librating with an amplitude of 36° and period of 1.75 yr. Thus, on the average the ratio of orbital periods relative to the longitude of apocenter is maintained at 3/4.

Two characteristics of this example are of note because, as we shall see, they are features of most orbit - orbit resonances. First, the stable configuration at conjunction is a "mirror configuration" (Roy and Ovenden, 1955). All velocity vectors are normal to a plane containing all the bodies. The reversibility of Newtonian mechanics implies that subsequent behavior is a mirror image of previous behavior. Second, the resonance tends to keep the satellites apart as conjunction is kept away from the longitude at which the two orbits are closest together.

## 3. ANALYSIS OF THE RESONANCE

The qualitative description, though accurate, would not be acceptable without quantitative confirmation. We need to find equations governing the behavior of conjunction of the two satellites relative to Hyperion's apocenter under the influence of Titan's perturbations. Clearly, an expression for the longitude of conjunction is crucial.

The longitude of conjunction of two satellites is a "stroboscopic" function of time: it is only meaningful at the instants of conjunction. But a continuous function connecting the stroboscopic points can be defined. In the Titan - Hyperion case the following definition is possible:

Longitude of conjunction $\equiv$ 4 × (longitude of Hyperion) −3 × (longitude of Titan)

When satellites' longitudes have the same value, this function also takes that value so this continous function matches the stroboscopic points. For satellites in resonance, the longitude of conjunction varies slowly compared to the mean motions. The coefficients 4 and −3 are selected so that the continuous function varies slowly near the 3/4 commensurability of periods.

The Titan - Hyperion resonance is characterized by libration of the conjunction longitude about Hyperion's apocenter. The mean longitude of Hyperion equals the true longitude at this point. This is nearly true for Titan, also, since its orbit has a low eccentricity. Thus, the resonance can be described by the statement that the "resonance variable", $\Theta$, defined as

$$\Theta \equiv 4\lambda_2 - 3\lambda_1 - \tilde{\omega}_2 \qquad (1)$$

librates about the value 180°. Here $\lambda$ is the mean longitude and $\tilde{\omega}$ is the longitude of pericenter. (See Appendix). In any pair of satellites, subscripts 1 and 2 refer to the inner and outer one respectively.

The analysis of any resonance reduces to a study of the behavior of its resonance variable through application of Lagrange's equations for the variation of orbital elements (e.g. Danby, 1962). The expressions for the behavior contain $R$, the "disturbing function". $R$ is the potential which describes the perturbing effect of one satellite on the Keplerian orbit of another:

$$R = Gm\left[\frac{1}{\Delta} - \frac{\vec{r}\cdot\vec{r}\,'}{r^3}\right]. \tag{2}$$

Here G is the gravitational constant, $m$ is the mass of the perturbing satellite, $\Delta$ is the distance between satellites, $\vec{r}$ is the position of a satellite with respect to the primary, and primed quantities denote the perturbed satellite. The first term in brackets represents the direct potential at the perturbed satellite due to the disturber. The second term represents indirect effects due to the force of the disturber exerted on the primary.

The disturbing function is customarily expanded into a Fourier series (e.g. Moulton, 1914; Brouwer and Clemence, 1961a; Kaula, 1966), the use of which facilitates analytic solution. The expansion takes the form

$$R/m = \sum C(e,\ e',\ i,\ i',\ a,\ a') \cos\left[q\,\lambda + q'\,\lambda' + j\,\tilde{\omega} + j'\,\tilde{\omega}' + p\Omega + p'\Omega'\right] \tag{3}$$

where summation is over the integers $q,\ q',\ j,\ j',\ p,\ p'$. $C$ is also a function of these integers. Orbital elements are defined in the Appendix. The following important mathematical properties of this expansion define it as a "d'Alembert series": $C$ is of the order $e^{|j|}e'^{|j'|}i^{|p|}i'^{|p'|}$ ; the sum $p + p'$ is even; and $q + q' + j + j' + p + p' = 0$. It follows that $|j| + |j'| + |p| + |p'| \geq |q + q'|$. The physical basis for these mathematical properties will become apparent as the intimate relation between the expansion and possible resonant modes is revealed.

To zeroth order in the disturbing mass, only the $\lambda$'s vary with time. Hence the terms are classified as "secular" if $q = q' = 0$, "long-period" if $q\,n + q'n'$ is small compared to either $n$ or $n'$, and "short-period" otherwise. Short-period terms are assumed ineffective because short periods do not allow large perturbations to build up, and, to at least first order in the disturbance, the effects time-average to zero over each short period. In the analysis of a resonance, "critical terms" which contain the resonance variable (or its multiples) as their argument, dominate the behavior. As should become apparent after we have considered an assortment of types of resonances, the expansion is also useful from another point of view: inspection of the various terms reveals the sorts of resonant interactions which could theoretically exist. Such insight is useful in evaluating the significance of the resonances which do occur in the solar system. Moreover, the intimate correspondence between the terms of the expansion and various modes of physical behavior should be borne in mind in seeking to understand the convergence properties of the series. It should be noted that the expansion discussed here definitely does not converge for the case $r = r'$.

For Titan and Hyperion, any term containing $\Theta$ as argument is "critical" and must be retained. Strictly speaking, any other term containing $q/q' = 3/4$ would also have a long period. However, to simplify our discussion we may make the following approximations: $e_1$, $i_1$ and $i_2$ are zero and $e_2$ is sufficiently small for $O(e_2{}^2)$ terms in the disturbing function to be neglected. For purposes of constructing an accurate ephemeris for Hyperion, these assumptions would be unjustified. But for our object, explanation of the resonance mechanism, this approximation is useful. By the d'Alembert properties these assumptions allow elimination of all terms except the secular ones and the $\Theta$-term. The $\Theta$-term takes the form $(Gm_1/a_2)e_2F \cos\Theta$, where $F$ is a positive function of $\alpha = a_1/a_2$, but $F$ is nearly constant ($\simeq 5$) for our purposes (Greenberg, 1973).

Variation of the orbital elements is given by Lagrange's planetary equations, which for the coplanar, small-eccentricity model take the form

$$\dot{n}_2 = -\frac{3}{a_2^2}\frac{\partial R}{\partial \lambda_2} = 12 e_2 \mu_1 n_2^2 F \sin\Theta \tag{4}$$

$$\dot{e}_2 = -\frac{1}{n_2 a_2^2 e_2}\frac{\partial R}{\partial \tilde{\omega}_2} = -\mu_1 n_2 F \sin\Theta \tag{5}$$

$$\dot{\tilde{\omega}}_2 = \frac{1}{n_2 a_2^2 e_2}\frac{\partial R}{\partial e_2} = \frac{\mu_1 n_2}{e_2} F \cos\Theta. \tag{6}$$

Here dots denote time derivatives and $\mu_1 = m_1/M$ where $M$ is the mass of the primary. Variation of $\varepsilon_2$, the mean longitude at epoch, is negligible. Note that Kepler's third law has been used in these evaluations. The variation of $\Theta$ is given by

$$\dot{\Theta} = 4n_2 - 3n_1 - \dot{\tilde{\omega}}_2 = 4n_2 - 3n_1 - \frac{\mu_1 n_2}{e_2} F \cos\Theta. \tag{7}$$

Equations (4), (5) and (7) define a set of trajectories in $e_2$, $n_2$, $\Theta$ space. Two modes of behavior are possible depending on the size of the eccentricity. For sufficiently small eccentricity, variation of $n_2$ is negligible, so behavior of $\Theta$ is dominated by $\dot{\tilde{\omega}}_2$, the last term in eq. (7). For larger eccentricities variation of $n_2$ dominates the behavior. For mass values appropriate to the Titan - Hyperion case, the latter mode operates for $e_2 \gtrsim 0.04$. Thus even in the larger-$e_2$ mode, our small-eccentricity forms for the disturbing function and the variation equations are meaningful.

One way to demonstrate the distinction between the two modes is to take the time derivative of eq. (7):

$$\ddot{\Theta} = 48 e_2 \mu_1 n_2^2 F \sin\Theta + \frac{\mu_1^2}{e_2^2} n_2^2 F^2 \cos\Theta \sin\Theta + \frac{\mu_1}{e_2} n_2 F \dot{\Theta} \sin\Theta. \tag{8}$$

In the larger-$e_2$ mode the first term dominates. Thus eq. (8) takes the form of the equation for a pendulum with stable equilibrium at $\Theta = 180°$ and unstable equilibrium at $\Theta = 0$. (Strictly speaking, the coefficients' dependence on $e_2$ and $n_2$ must be taken into account (e.g. Greenberg, 1973a). Near the equilibrium values eqs. (4) and (5) show that $e_2$ and $n_2$ vary slowly.) This behavior is in perfect agreement with the qualitative description of the resonance mechanism of Section 2 and with the observed behavior of Titan and Hyperion. The mechanism requires significant eccentricity and involves variation of $n_2$ with negligible rotation of the orbital major axis.

## 4. SMALL ECCENTRICITY MECHANISM

For sufficiently small $e_2$ the second term in the expression for $\ddot{\Theta}$ would dominate. The second term is $\dot{e}_2(\partial\dot{\tilde{\omega}}_2/\partial e_2)$ and thus represents acceleration of $\dot{\tilde{\omega}}_2$ due to changes in $e_2$. The small-$e_2$ mechanism clearly differs from the larger-$e_2$ mechanism in several respects:

(i) Rotation of the line of apsides plays a significant role because $\dot{\tilde{\omega}}_2 \propto 1/e_2$. This dependence on $e_2$ is physically reasonable, because a nearly-circular orbit has a weakly defined line of apsides that can be reoriented by the slightest perturbing force

(ii) Variation of mean motion is less significant because $\dot{n}_2 \propto e_2$. The physical mechanism described in Section 2 required a significant eccentricity to explain variation of $n_2$.

(iii) The $\Theta$-dependence of the second term in eq. (8) indicates stable equilibria at $\Theta = 0^\circ$ and $180^\circ$.

Behavior in the small-$e_2$ case can be studied by considering variation of the quantities $h \equiv e_2 \cos \Theta$ and $k \equiv e_2 \sin \Theta$ via the chain rule:

$$\dot{h} = -k[4n_2 - 3n_1] \tag{9}$$

$$\dot{k} = +h[4n_2 - 3n_1] - \mu_1 n_2 F \ . \tag{10}$$

For sufficiently small $e_2$ the mean motions are nearly constant. Thus the solution to (9) and (10) is

$$h = C \cos\{[4n_2 - 3n_1]t + \delta\} + A \tag{11}$$

$$k = C \cos\{[4n_2 - 3n_1]t + \delta\} \tag{12}$$

where $C$ and $\delta$ are arbitrary constants of integration and $A \equiv \mu_1 n_2 F/(4n_2 - 3n_1)$. This solution can be represented in $(h,k)$ rectangular coordinates, equivalent to $(e,\Theta)$ polar coordinates, as motion in a circle of radius $C$ about the point $h = A$, $k = 0$ at rate $4n_2 - 3n_1$ (Fig. 2).

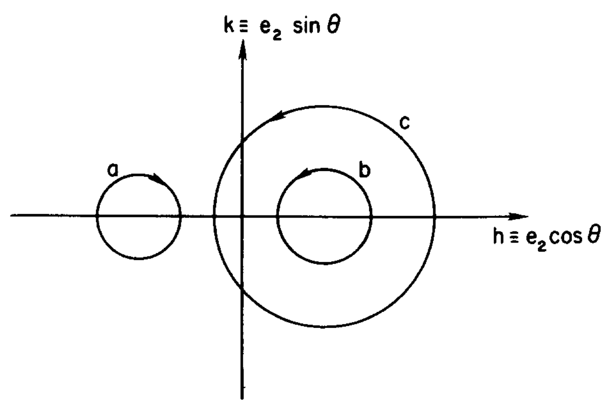


Fig. 2. Solutions to the variation equations for the small eccentricity case. In case a, $n_2/n_1 < 3/4$ and $C < |A|$, so $\Theta$ librates about $180^\circ$. In case b, $n_2/n_1 > 3/4$ and $C < |A|$ so $\Theta$ librates about $0^\circ$. In case c, $n_2/n_1 > 3/4$ but $C > |A|$ so $\Theta$ circulates.

Note that the sign of $A$ depends on the sign of $(n_2/n_1) - (3/4)$. For $C > |A|$, $\Theta$ circulates; for $C < |A|$, $\Theta$ librates about $0^\circ$ (if $A > 0$) or $180^\circ$ (if $A < 0$). Note also that this small-$e_2$ mechanism cannot operate with the ratio $n_2/n_1$ too near to the exact commensurability. In such a case $A$ would be large, so that $e_2$ could not remain small. $C$ is generally called the "free" eccentricity because it is an arbitrary parameter of the motion, while $|A|$ is the irreducible eccentricity "forced" by $m_1$.

The small-$e_2$ mechanism can be described physically as well as mathematically. The effect of $m_1$ on $m_2$ can be approximated by a radial impulse applied at conjunction. (There are tangential forces as well, but with both orbits nearly circular these forces reverse at conjunction so that their net effect is much less important.) The effect of such an impulse on the longitude of pericenter and on $e_2$ varies as cos $\eta$ and -sin $\eta$, respectively, where $\eta$ is the true anomaly at which the impulse is exerted. Applied at apocenter, such an impulse causes the major axis to regress; applied at pericenter, it causes the major axis to advance. The maximum decrease in $e_2$ is produced by an impulse exerted 90° before apocenter; the maximum increase is produced by an impulse 90° after apocenter.

Suppose conjunction occurs at apocenter with a libration amplitude of zero. Since the radial impulse causes apocenter to regress, $n_2/n_1$ must be less than 3/4 to ensure that the conjunction longitude regresses to keep pace with apocenter. Such a situation would be represented in our analysis by the case $A < 0$, $C = 0$ (a point on the negative $h$ axis in Fig. 2).

If the conjunction longitude does not regress quite as fast as apocenter and if conjunction initially occurs soon after $m_2$'s apocenter passage, then the radial impulses will cause $e_2$ to increase and the regression of the major axis to decrease correspondingly. Conjunction can then overtake apocenter. Similarly, wherever conjunction occurs, the nearest apse is accelerated toward it. The longitude of conjunction can be stable at either apocenter or pericenter. The mechanism involves acceleration of apsides toward the longitude of conjunction through variation of eccentricity. In contrast, the higher-eccentricity mechanism involves acceleration of the longitude of conjunction toward apocenter through variation of $n_2$.

Like the higher-eccentricity mechanism, the low-eccentricity mechanism is stable in "mirror configurations". The stable configurations do not, however, necessarily prevent conjunction where the orbits are closest together. On the other hand, this latter property is not particularly significant for nearly circular orbits.

No example of this 3/4 small-eccentricity resonance is known in the present solar system. One theory of the origin of the Titan - Hyperion resonance (Greenberg, 1973a) suggests that capture from original circulation occurred when $e_2$ was small. Subsequent increase of the ratio of $n_2/n_1$ (perhaps due to tidal evolution of Titan's orbit or drag effects in an early environment) toward its present value just less than 3/4 would increase $|A|$ until the present high-eccentricity mode became dominant.

## 5. ENCELADUS-DIONE RESONANCE

The resonance between Enceladus and Dione is an example of a small-eccentricity type resonance, but it differs in several important respects from the resonance described in the previous section (Woltjer, 1922a): (i) The ratio of orbital periods is nearly 1/2. (ii) Conjunction $(2\lambda_2 - \lambda_1)$ is locked to pericenter of the inner satellite, rather than to an apse of the outer one. (iii) The masses and eccentricities of the two satellites are comparable (See Table 1). Despite these differences, the concepts introduced in the previous sections are applicable. For a 1/2 commensurability, conjunction occurs at a single, fixed longitude. Thus a similar physical description holds for this case as for the previous one. It is necessary to be somewhat more careful in constructing a realistic model because of property (iii).

In our model we may assume coplanar motion with small non-zero eccentricities. First consider the effect of Dione, the outer one of the pair, on Enceladus. For very small eccentricities we know that, aside from small, short-period variations, the mean motions are virtually constant. However, the radial force exerted outward at each conjunction affects $\tilde{\omega}_1$ and $e_1$ in a manner analogous to the variation of eccentricity and longitude of pericenter in the previous example. As in that case, the nearest apse is accelerated towards the longitude of conjunction. In this manner the stable configuration with conjunction occuring at Enceladus' pericenter is maintained. Mathematically, the effect of $m_2$ on $m_1$ could be treated by retaining only long-period terms and these to lowest order in eccentricity, yielding $R$ of the form

$$R = (Gm_2/a_2)[F_A e_1 \cos \sigma_A + F_B e_2 \cos \sigma_B] ; \tag{13}$$

where $\sigma_A \equiv 2\lambda_2 - \lambda_1 - \tilde{\omega}_1$ and $\sigma_B \equiv 2\lambda_2 - \lambda_1 - \tilde{\omega}_2$. The variation equations for orbital elements of $m_1$ involve partial derivatives with respect to elements of the inner orbit. The second term in $R$ only plays a role in variation of $n_1$. But, for sufficiently small eccentricities, variation of $n_1$ becomes negligible compared to variation of $e_1$ and $\tilde{\omega}_1$. Thus the second term in $R$ plays an insignificant role (for our purposes) and the analysis becomes analogous to that of the previous section in which only one term was important. The value of $\sigma_A$ can librate about zero as observed (property (ii)) or, given suitable initial conditions, $\sigma_A$ could librate about 180°.

It is worth noting the physical significance that the second term would have in a higher-eccentricity case. To compute the energy exchange near conjunction as in the mechanism discussed in Section 2, it would be important to consider the location of conjunction relative to the apsides of both orbits.

The effect of $m_1$ on $m_2$ could be computed in a manner similar to that of $m_2$ on $m_1$. We find that conjunction could be stable at either apse of $m_2$, i.e. $\sigma_B$ could librate about 0° or 180°.

Why can't $\sigma_A$ and $\sigma_B$ both librate? In other words, given the observed and well-understood lock of Enceladus' pericenter to the longitude of conjunction, why isn't an apse of Dione's orbit also locked to conjunction? The answer to this question involves the effects of the oblateness of Saturn's figure on the satellites' orbits. The primary effect of the oblateness in this case is the precession of the line of apsides. Thus there is a tendency for $\tilde{\omega}_1$ and $\tilde{\omega}_2$ to increase secularly, each at its own rate dependent strongly on its distance from Saturn. In order to maintain the observed resonance lock, the two satellites must have a rate of precession of their longitude of conjunction (corresponding to the ratio of their mean motions) which is sufficiently close to the rate of precession of $\tilde{\omega}_1$. Otherwise, like a pendulum with too much kinetic energy, libration cannot occur. We do observe that the precession rates of conjunction and of $\tilde{\omega}_1$ are indeed sufficiently matched. However, the very different precession rate of $\tilde{\omega}_2$ due to Saturn's oblateness precludes its libration about conjunction.

One outstanding property of the Enceladus - Dione resonance is the very small amplitude of libration of $\sigma_A$ about 0 (Table 1). This small value implies either remarkable random initial conditions for the system or some evolutionary mechanism that has tended to lower the amplitude to its present value (Sinclair, 1972).

## 6. MIMAS - TETHYS RESONANCE

The oblateness of Saturn restricts the resonance behavior of satellites Enceladus and Dione. In describing the Titan - Hyperion resonance, I ignored the oblateness completely. This neglect can be justified partially by the reduced influence of oblateness at greater distance from Saturn. But, what is more important, any oblateness effects can be incorporated into the Titan - Hyperion model with no change in the qualitative nature of the mechanism. The oblateness simply implies that the stable configuration requires a slightly different ratio of mean motions to compensate for the additional motion of Hyperion's apsides. A third resonance in Saturn's satellite system, the Mimas - Tethys interaction, involves the planet's oblateness as an essential part of the mechanism.

The resonances described so far are of the eccentricity-type: the conjunctions librate about an apse of one orbit. The Mimas - Tethys resonance is of the inclination-type; these satellites have an orbital period of 1/2 (see Table 1) and their conjunction librates about the midpoint between their two ascending nodes on Saturn's equatorial plane. The classical analysis of eccentricity-type resonances has suggested the physical interpretation which reveals the underlying mechanism. In the case of the inclination resonance, the search for an analogous physical interpretation reveals features of the resonance which are obscured by the classical theory.

This classical analysis, developed by H. Struve (Tisserand, 1896) and applied to an evolution model by Allan (1969), is a study of the behavior of the resonance variable,

$$\psi \equiv 4\lambda_2 - 2\lambda_1 - \Omega_2 - \Omega_1 . \tag{14}$$

$\psi$ can be interpreted as twice the difference between the conjunction longitude, $\xi \equiv 2\lambda_2 - \lambda_1$, and the average longitude of the nodes, $\Omega_{avg} \equiv (\Omega_1 + \Omega_2)/2$. In the analysis, $\ddot{\psi}$ is evaluated by Lagrange's planetary equations. If terms are retained only to lowest order in inclinations and in satellite-to-primary mass ratios, and if short-period terms are neglected, then

$$\ddot{\psi} = - C i_1 i_2 \sin \psi \tag{15}$$

where the coefficient $C > 0$ is a function of the semi-major axes. It follows from (15) that $\psi$ is stable at 0, i.e. conjunction is stable at $\Omega_{avg}$.

As outlined in the previous paragraph, the analysis appears to hold for any reference plane from which the inclinations are small, not only for the equatorial plane. But changing the reference plane slightly can drastically alter $\Omega_{avg}$. Why does the Mimas - Tethys conjunction librate about $\Omega_{avg}$ measured on the equatorial plane instead of on some other reference plane? From the point of view of the satellites' dynamics, the equatorial plane is special because of Saturn's oblateness and, in fact, this characteristic is included implicitly in the traditional analysis (Greenberg, 1973b).

Some of the short-period terms that were neglected from the disturbing function contained the following as arguments of cosines:

$$\psi_A = 4\lambda_2 - 2\lambda_1 - 2\Omega_1 \tag{16}$$

and

$$\psi_B = 4\lambda_2 - 2\lambda_1 - 2\Omega_2 . \tag{17}$$

The reason that these terms have significantly shorter periods than the $\psi$-term (on the order of a year compared with a libration period of ~70 yr) is that the longitudes of the nodes precess rapidly due to the planet's oblateness. Even with this precession, the periods of these terms are much longer than those of the other neglected short-period terms. (The latter are on the order of orbital periods or about a day.) In order to understand the mechanism that maintains the Mimas - Tethys resonance, we need to answer two interrelated questions: What is the effect of terms with arguments $\psi_A$ and $\psi_B$? What role does the oblateness play? The following analytic approach avoids obscuring these points.

We can begin to answer these questions by first investigating the properties of inclination-type resonances with a model that does not include oblateness. We consider two small satellites in circular orbits with orbital inclinations small relative to an inertial reference plane and with periods nearly commensurable in a ratio of 1/2. The appropriate disturbing function at either satellite is

$$R = Gm[C_A i_1^2 \cos \psi_A - C i_1 i_2 \cos \psi + C_B i_2^2 \cos \psi_B], \tag{18}$$

where $m$ is the mass of the other satellite (the disturber) and the coefficients $C$, $C_A$ and $C_B$ are positive functions of the semi-major axes.

The resonance analysis would be difficult with the disturbing function containing terms with three different arguments instead of one. This problem can be circumvented by specifying that one satellite is so much more massive than the other that its orbit plane is virtually fixed. If this fixed plane is taken as the reference plane, then the inclination of the more massive satellite's orbit is zero and two terms in (18) can be eliminated.

If the masses are comparable, as in the case of Mimas and Tethys (see Table 1), we need another way to reduce the disturbing function to a single argument. We assume that the satellites' total orbital angular momentum, $\vec{L}$, measured relative to the planet is constant. Then we select as a reference plane the plane through the center of the planet and normal to $\vec{L}$. Since $\vec{L}$ and the individual orbital angular momentum vectors are coplanar, $\Omega_1 = \Omega_2 + 180°$ in this reference frame. It follows that

$$\cos \psi_A = \cos \psi_B = -\cos \psi \tag{19}$$

and $R$ contains only one argument:

$$R = -Gm[C_A i_1^2 + C i_1 i_2 + C_B i_2^2] \cos \psi. \tag{20}$$

In order to determine stable values of $\psi$ we must evaluate

$$\ddot{\psi} = 4\dot{n}_2 - 2\dot{n}_1 + 4\dot{\varepsilon}_2 - 2\dot{\varepsilon}_1 - \ddot{\Omega}_1 - \ddot{\Omega}_2. \tag{21}$$

Evaluating the variation of mean motions by Lagrange's equations gives

$$\ddot{\psi} = -12g(C_A i_1^2 + C i_1 i_2 + C_B i_2^2) \sin \psi + 4\ddot{\varepsilon}_2 - 2\ddot{\varepsilon}_1 - \ddot{\Omega}_1 - \ddot{\Omega}_2, \tag{22}$$

where $g = \mu_2 n_1^2 a_1 + 4\mu_1 n_2^2 a_2$. Following Allan (1969), to first order in $\mu$, we can ignore $\ddot{\varepsilon}_1$, $\ddot{\varepsilon}_2$ and variation of orbital elements as they appear in the coefficient of $\sin \psi$. In general, we cannot ignore $\ddot{\Omega}_1$ and $\ddot{\Omega}_2$ because, although they are of higher order in $\mu$, they are of lower order in $i$. According to Lagrange's equations,

$$\dot{\Omega}_1 = \mu_2 n_1 a_1 [(i_2/i_1) C + 2C_A] \cos \psi. \tag{23}$$

Thanks to our choice of reference plane, $i_2/i_1$ is a function of $a_2/a_1$ and $\ddot{\Omega}_1 = \ddot{\Omega}_2$. Thus

$$\ddot{\Omega}_1 + \ddot{\Omega}_2 = 2\ddot{\Omega}_1 = 2\mu_2 n_1 a_1 [(i_2/i_1)C + 2C_A] \dot{\psi} \sin \psi + O(\mu^2 i^2). \tag{24}$$

The complete expression for $\ddot{\psi}$ from (22) and (24) is, to our approximation,

$$\ddot{\psi} = -A \sin \psi + B\dot{\psi} \sin \psi, \tag{25}$$

where $A$ and $B$ are positive constant coefficients. This expression shows that $\psi$ is stable at 0. This condition, unlike that of the Mimas - Tethys case, can be described in a manner independent of the reference plane. Conjunction is stable at either of the two longitudes 90° from the satellites' "mutual nodes" (the nodes of one orbit on the plane of the other). As in the larger-eccentricity resonances the stable condition is a mirror-configuration with conjunction stable at that longitude where the orbits are farthest apart.

These characteristics can be interpreted qualitatively by a mechanism analogous to the higher-eccentricity mechanism which governs the Titan - Hyperion resonance. Suppose the orbital mean motions are in a ratio of 2/1 so that conjunction occurs repeatedly at the same meridian within 90° after the satellites' mutual node as shown in Fig. 3. Since the mutual inclination is small, virtually no orbital energy is exchanged at conjunction by gravitational interaction.

However, energy *is* exchanged before and after conjunction. It is evident from Fig. 3 that the gravitational effects are greater before conjunction because the satellites are closer to one another than they are after conjunction. Moreover, the lines of force are directed more nearly along the direction of motion before than after conjunction.

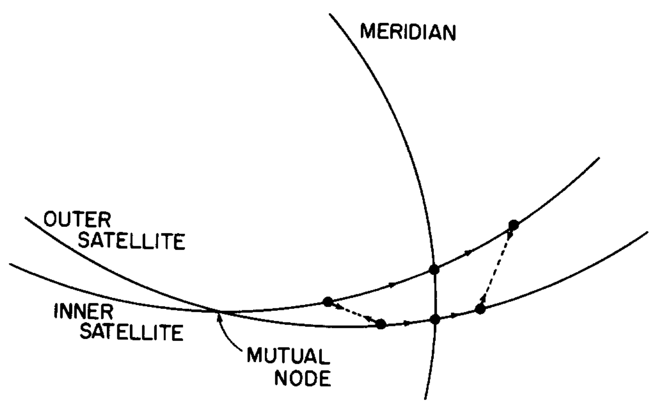


Fig. 3. Orbits of two satellites projected onto the celestial sphere. At conjunction they both lie on the same meridian. Mutual forces just before and just after conjunction are also shown. These forces control the net energy exchange which can maintain a commensurability.

Thus the net effect of the gravitational interaction is to transfer energy from the outer satellite to the inner one as occurs before conjunction. This energy transfer slows the angular velocity of the inner satellite, lowering the mean motion ratio below the commensurable value 2/1. Thus the next conjunction occurs further away from the mutual node of the satellites. Similarly if conjunction occurs before the mutual node, the net effect of the gravitational interaction is to cause conjunction to regress away from the mutual node. Since there are two mutual nodes 180° apart, conjunction is stable at either of the two longitudes 90° from the satellites' mutual nodes.

For eccentricity resonances, if the eccentricities are very small, this sort of mechanism is weak. However, the orientation of the major axis is more readily varied when the eccentricity is small, and the radial perturbation forces tend to maintain the alignment of an apse with the conjunction longitude. For inclination resonances, the expression for $\ddot{\psi}$ contains no term that maintains stability for extremely low inclinations. (Compare eq. (25) with eq. (8)). The physical reason is that while the orientations of the node lines are more readily varied when the inclination is small, the normal perturbation forces (which tend to vary the orientations) decrease with the mutual inclination. There is no low-inclination resonance mechanism analogous to the small-eccentricity mechanism.

This last point demonstrates the physical importance of the d'Alembert properties of the disturbing function. These properties rule out any term in the disturbing function of first order in inclination. Thus they effectively rule out any physically impossible low-inclination mechanism.

Now that we have reviewed the traditional analysis of the Mimas - Tethys case with substantial oblateness of the primary and have examined the properties of inclination resonance with a spherical primary, we may continue our analysis of the relative roles of oblateness and of mutual perturbations by examining the intermediate case: the case with moderate oblateness where the nodes precess significantly, but not so fast that $\psi_A$ and $\psi_B$ can be considered to be short-period arguments. In order to complete our study of the interaction of oblateness and mutual perturbations we shall then consider the Mimas - Tethys resonance as a limit of this more general case.

All orbital elements are referred to the equatorial plane of the central body. Evaluation of $C$, $C_A$ and $C_B$ yields

$$C = 2C_A = 2C_B \simeq 0.4/a_2. \tag{26}$$

With the definition $\Delta\Omega \equiv \Omega_2 - \Omega_1$, $R$ takes the form

$$R/Gm = \frac{C}{2} i_1{}^2\cos (\psi + \Delta\Omega) - Ci_1 i_2\cos \psi + \frac{C}{2} i_2{}^2\cos (\psi - \Delta\Omega). \tag{27}$$

From Lagrange's equations variation of the conjunction longitude, $\xi \equiv 2\lambda_2 - \lambda_1$, is governed by

$$\ddot{\xi} = 6g\left[\frac{C}{2} i_1{}^2 \sin (\psi - \Delta\Omega) - Ci_1 i_2\sin \psi + \frac{C}{2} i_2{}^2 \sin(\psi - \Delta\Omega)\right]. \tag{28}$$

Due to the primary's oblateness, $\Omega_1$ and $\Omega_2$ are significantly varying functions of time. However, at any instant, there are "quasi-equilibrium" values of $\xi$ at which $\ddot{\xi} = 0$. Those quasi-equilibrium values toward which the value of $\xi$ is accelerated shall be called "quasi-stable", the others "quasi-unstable". If (28) is set equal to zero and rearranged appropriately, the quasi-equilibrium values can be expressed by

$$\xi = \Omega_1 + \tan^{-1}\left[\frac{-\sin \Delta\Omega}{(i_1/i_2) - \cos \Delta\Omega} + \frac{\ell\pi}{2}\right] \tag{29}$$

where $\ell$ is any integer. It can readily be verified that solution (29) gives quasi-unstable values for even $\ell$ and quasi-stable values for odd $\ell$. By straightforward spherical geometry, for even $\ell$, $\xi$ as described by (29) is identical to the longitudes of the mutual nodes (Roy and Ovenden, 1955); for odd $\ell$, it equals the longitudes 90° away. We conclude that for inclination-type resonances with substantial primary oblateness, as well as for those with spherical primaries, the mutual perturbations of the satellites tend to draw the longitude of conjunction toward those longitudes 90° from the satellites' mutual nodes.

Why, then, is the Mimas - Tethys conjunction longitude observed to librate about $\Omega_{avg}$ measured on Saturn's equatorial plane, rather than about one of the quasi-stable longitudes? In order to answer this question, we next consider the behavior with time of the quasi-stable longitudes and of $\Omega_{avg}$.

In Fig. 4, $\Omega_{avg}$ (measured from $\Omega_1$) is shown as a function of $\Delta\Omega$ given by

$$\Omega_{avg} - \Omega_1 = \frac{\Delta\Omega}{2} + N\pi \tag{30}$$

where $N$ is any integer. The independent variable $\Delta\Omega$ is convenient because it varies monotonically and, for the Mimas - Tethys case, nearly linearly with time.

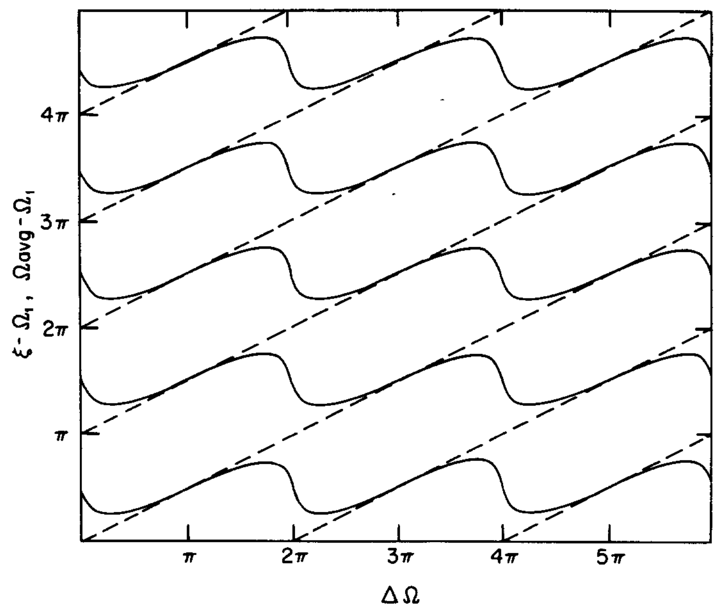


Fig. 4. $\Omega_{avg}$ (measured from $\Omega_1$) shown by dotted lines, and quasi-stable longitudes (also measured from $\Omega_1$), shown by solid curves, as multi-valued functions of $\Delta\Omega$ (from Greenberg, 1973b).

Figure 4 also shows, as functions of $\Delta\Omega$, the quasi-stable longitudes (measured from $\Omega_1$) given by eq. (29) with odd $\ell$. Here the inclinations have been chosen to have the observed values for Mimas and Tethys. If $i_1/i_2$ had a value closer to unity, then the "dull saw-tooth" curves would be more "sharp". For the singular case $i_1 = i_2$, the saw-tooth curve would be so sharp that the quasi-stable longitudes would equal the average longitudes of the ascending nodes except at the dincontinuities at $\Delta\Omega = 2N\pi$ where mutual nodes would not be well-defined. Our problem would be solved because in this case the quasi-stable longitudes would always be at the midpoint between equatorial nodes. However, in the more realistic case where the inclinations are not equal, the curves of Fig. 4 represent the quasi-stable longitudes. Quasi-stable longitudes oscillate about fixed values $\pi/2 + N\pi - \Omega_1$ while values of $\Omega_{avg}$ circulate through 360° relative to $\Omega_1$.

Suppose conjunction occurs at the midpoint between the equatorial nodes at an instant when that longitude is quasi-stable (e.g. at $\Delta\Omega = \pi$). At the quasi-stable longitude the mutual gravitational interaction does not accelerate conjunction one way or the other. Now suppose the ratio of the mean motions of the two satellites is such that the conjunction longitude follows one of the dotted lines. Immediately the mutual interaction tends to pull conjunction down toward the quasi-stable longitude. However, the time scale for response to this pull is quite long. (The libration period for the Mimas - Tethys case is about 70 yr while $\Delta\Omega$ circulates in ~1 yr.) Before conjunction can respond significantly it has moved along the dotted line to a region where the mutual interaction tends to pull conjunction up toward the nearest quasi-stable longitude. As conjunction follows a dotted line, it experiences exactly as much pull up as down, so that the net effect is zero. In this sense, we see that conjunction at the midpoint of the equatorial nodes is an equilibrium configuration.

Is this a stable configuration? Suppose conjunction moves just below a dotted line. Most of the time it is pulled upwards toward the quasi-stable longitude so that the net effect over several years is to draw conjunction toward the dotted line. In this way we can show heuristically that conjunction is in stable equilibrium on the dotted line.

In summary, we have found that the instantaneous influence of mutual perturbation is to draw conjunction toward the quasi-stable longitudes, but that the average effect, due to the planet's oblateness, is to draw conjunction toward the *observed* stable longitudes. Thus, this figure helps to demonstrate how the mutual perturbations (which draw conjunction toward longitudes $90^\circ$ from the mutual nodes) and the oblateness perturbations (which precess the nodes) can combine to produce the observed Mimas - Tethys resonance.

The mathematical theory of this resonance indicates that conjunction might also be stable at either satellite's *descending* node on Saturn's equator, i.e. that $\psi_A$ or $\psi_B$ might librate about zero if Mimas and Tethys were given suitable mean motions (Sinclair, 1972). These alternate possibilities are analogous to the alternatives available in the Enceladus - Dione case which depended on the exact mean motion ratio. In $\psi_A$ libration, conjunction (measured from $\Omega_1$) would librate about $\pi/2 + N\pi$. In terms of our figure, conjunction would librate about a horizontal straight line. Along such a line the time-averaged pull toward quasi-stable longitudes is zero. Similarly, in $\psi_B$ libration, conjunction (measured from $\Omega_1$) would librate about $\Delta\Omega + N\pi + \pi/2$, a line about which the time-averaged pull is again zero.

One additional interpretation of the figure may be useful. We can regard the solid curves as representing, at any instant, the positions of minima on a sinusoidal potential field governing the behavior of the conjunction longitude. As time goes on, the potential topography alternately follows the average nodes and then jumps back $180^\circ$ quite suddenly. In the case of the observed Mimas - Tethys resonance, the sudden jumps occur too quickly for the system to respond.

The stable configuration in the observed Mimas - Tethys resonance is not a mirror configuration. However, Roy and Ovenden (1955) suggested that the period of libration of conjunction about $\Omega_{avg}$ might be commensurable with the period of circulation of $\Delta\Omega$ in such a way that conjunction might periodically return to mirror-configurations at points like $(\pi,\pi/2)$ and $(3\pi,3\pi/2)$ in our figure. We have seen that the mirror configurations play an even more profound role as they determine the quasi-equilibrium longitudes.

## 7. NEPTUNE - PLUTO RESONANCE

Neptune and Pluto are involved in a resonance which incorporates features of both eccentricity- and inclination-type resonances although, strictly speaking, none of the mechanisms described so far are applicable to this case. These planets have mean motions near a ratio of 3 : 2 with, as Cohen and Hubbard (1965) discovered through numerical integration, conjunction librating about Pluto's aphelion longitude with an amplitude of nearly $80^\circ$ and a period of nearly $2\times10^4$ yr (Table 1). Since Pluto has a large orbital eccentricity (0.25) and negligible mass compared to Neptune, this resonance is superficially reminiscent of the Titan - Hyperion case. However, Pluto also has a high inclination ($17^\circ$). Williams and Benson (1971) have demonstrated that Pluto's argument of perihelion (defined in the Appendix) librates about $90^\circ$ with an amplitude of about $24^\circ$ and period of $4\times10^6$ yr. It follows that conjunction is locked to a longitude $90^\circ$ from the mutual nodes of the two orbits, a reasonable configuration according to the considerations of the previous section.

This resonance demonstrates both of the common properties of resonances. The stable conditions are mirror configurations. Moreover, they represent configurations which tend to avoid close approaches of the two bodies. In this resonance the latter property is particularly significant, because Pluto's eccentricity is so large that at perihelion it is closer to the sun than Neptune (Fig. 5). Before the resonance was discovered, it was quite reasonable to expect a collision or near-collision in the future or to suggest that such an event had occurred in the past (Lyttleton, 1936). The resonance precludes such events unless some dissipative forces which have not been considered play a role in the orbital history.

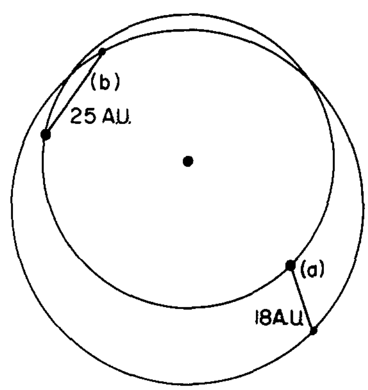


Fig. 5. Orbits of Neptune and Pluto: (a) Planets at minimum separation near conjunction. (b) Planets at next minimum separation, which is nowhere near conjunction, but is comparable to separation in (a). (From Cohen and Hubbard, 1965).

Cohen and Hubbard gave a qualitative explanation of the eccentricity aspects of this resonance which is rather different than the mechanism presented in Section 2 for the Titan - Hyperion case. The Titan - Hyperion case fails in this case because it depended on the notion that the dominant effect of Titan on Hyperion was due to forces exerted near conjunction where the bodies' separation was minimal. In the Neptune - Pluto case, however, Pluto's eccentricity is so large that secondary minima are reached far from conjunction. For example, Fig. 5 shows a case where a minimum separation near conjunction is followed by a comparable minimum far from conjunction. The mathematical analysis of the Titan - Hyperion resonance, like the physical model, breaks down in the Neptune - Pluto case. For $e_2 \gtrsim 0.25$, terms from the disturbing function of lowest order in eccentricity are not adequate; high-order terms must be retained. According to the d'Alembert rules these additional terms represent higher order harmonics of the Fourier expansion, which correspond to the additional periodic minimal approaches. These changes complicate the analysis, as does an additional mathematical problem: the coefficients of the traditional expansion are ill-defined at those instants when both planets are equally distant from the sun. As a result of such difficulties, numerical studies of this resonance have generally been more useful than analytic ones.

Cohen and Hubbard described the qualitative physical mechanism of Neptune - Pluto resonance by considering the orbit of Pluto in a reference frame rotating with Neptune's mean motion (Fig. 6). Suppose conjunction occurs somewhat after Pluto's aphelion passage, but before perihelion with a ratio of orbital periods of 3 : 2. Such a configuration would be represented with Neptune in the left-most position in Fig. 6. Now consider the effect of Neptune on Pluto's orbital energy averaged over a synodic period. The greatest effect is when Pluto is on the left-most perihelion loop. Here the pull of Neptune is large because the planets are closer together than on most of the path. Moreover, the orbital energy exchange is enhanced at this point because the force makes a large angle with the Sun - Pluto line. This dominant force is in the direction of Pluto's motion in a non-rotating frame so the outer planet gains energy and the ratio of orbital periods rises above 3 : 2. In this way conjunction is accelerated back towards aphelion. Such restoring accelerations govern the libration about aphelion.

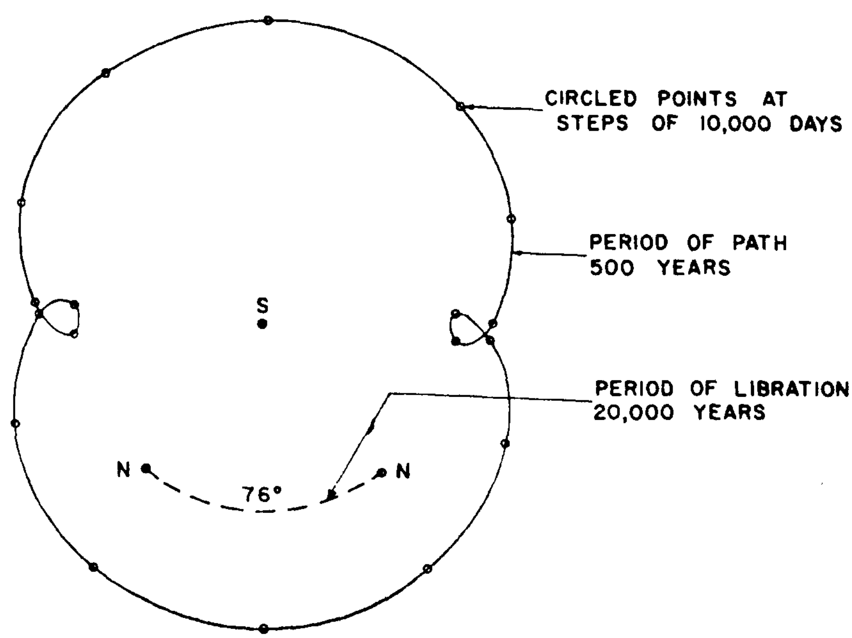


Fig. 6. Path of Pluto in a reference frame rotating with Neptune's mean motion. Because of the commensurability of orbital periods, Pluto makes two complete orbits around the sun between conjunctions with Neptune. Thus there are two perihelia and two aphelia on Pluto's path in this frame. The perihelia appear as loops, because at perihelia Pluto's angular velocity with respect to the sun is greater than Neptune's. Libration is represented by oscillation of the position of Neptune in this frame. (From Cohen and Hubbard, 1965).

Given that conjunction is locked to aphelion in this way, conjunction cannot also be locked 90° from the ascending node unless aphelion is maintained there. In fact the argument of perihelion is maintained at a value of 90° by a mechanism independent of the commensurability. Kozai (1962) has shown that under certain circumstances the argument of perihelion of an asteroid under the influence of Jupiter can be locked to a value of 90° when only secular parts of the disturbing function are involved. Cincinnati (1373) is an example of such an asteroid. Pluto's orbit is apparently maintained in a similar lock. Thus the inclination aspect of the Neptune - Pluto resonance, although superficially similar to the Mimas - Tethys case, is in fact governed by a very different mechanism.

## 8. SECULAR RESONANCES

Resonances which involve only secular terms are now believed to play an important role in the dynamics of the solar system (Williams, 1973). These resonances do not involve commensurabilities of orbital periods. Thus aside from secular terms, no long-period terms need be retained in the disturbing function. The theory leads to oscillatory behavior of certain elements which belies the commonly applied term "secular resonance".

One possible effect of a secular resonance might be maintenance of an alignment of the orientation of the major axes of two or more satellites. Such a stable alignment does exist between Saturn's satellites Rhea and Titan (Woltjer, 1922b). Rhea's longitude of pericenter librates around Titan's with an amplitude of 9°.5 and a period of 38 yr (Struve, 1928). Brouwer and Clemence (1961a) and Hagihara (1972) state that this case is not a libration. They point out that the orbital periods are not commensurable and the mean longitudes do not librate. However, use of the term "libration" is appropriate in this case provided that it is clear what is librating. Study of the Titan - Rhea interaction provides insight into the nature of secular resonance, although we shall see that the term "resonance" cannot apply to this particular case.

Rhea's behavior is dominated by the effects of Titan and of Saturn's oblate figure. For our purposes the effects of other satellites are negligible. Titan's orbit can be regarded as unperturbed except for a slow uniform precession of its apsides due to the combined effects of the oblate planet, the other satellites and the Sun. All motion will be assumed to be equatorial. Let us consider the possible behavior of such a model (Woltjer, 1922b).

The disturbing function at the inner satellite, without terms of short period and of high order in eccentricity, is

$$R = \tfrac{1}{2}Jr^2GMa_1^{-3}e_1^{\,2} + (Gm_2/a_2)F_x(\alpha)\, e_1^{\,2} - (Gm_2/a_2)F_y(\alpha)e_1 e_2\cos\phi. \tag{31}$$

Here $J$, $r$ and $M$ are the planet's oblateness coefficient, radius and mass, respectively; $\alpha \equiv a_1/a_2$; $\phi \equiv \tilde{\omega}_1 - \tilde{\omega}_2$; the values of $F_x$ and $F_y$ vary monotonically from $\sim 10^{-2}$ for $\alpha = 0.2$ to $\sim 1$ for $\alpha = 0.8$. The first term is due to the planet's oblateness; the second represents the effect of $m_2$ distributed around a ring of radius $a_2$. Note that, since variation of $a_1$ and $a_2$ is negligible, the "oblateness term" and the "ring term" have the same functional form, as one might expect intuitively. The last term is the "critical" term. It is important when the precession rates are nearly identical, because then $\phi$ is nearly constant.

Because Rhea's eccentricity is very small ($\sim 0.001$), we put $R$ in terms of $h$ and $k$, redefined as $h \equiv e\cos\tilde{\omega}$ and $k \equiv e\sin\tilde{\omega}$:

$$R = [\,\tfrac{1}{2}Jr^2GMa_1^{-3} + (Gm_2/a_2)F_x]\,(h_1^{\,2} + k_1^{\,2}) - [\,(Gm_2/a_2)F_y]\,(h_1h_2 + k_1k_2). \tag{32}$$

For small $e$ and low inclination, $h$ and $k$ are governed by variation equations derived from eqs. (5) and (6):

$$\frac{dh}{dt} = -\frac{1}{na^2}\frac{\partial R}{\partial k} \text{ and } \frac{dk}{dt} = \frac{1}{na^2}\frac{\partial R}{\partial h}\,. \tag{33}$$

Inserting $R$ from (32) into eq. (33) and applying Kepler's third law yields

$$\frac{dh_1}{dt} = -Ak_1 + Bk_2 \text{ and } \frac{dk_1}{dt} = Ah_1 - Bh_2 \tag{34}$$

where

$$A \equiv n_1[\,Jr^2a_1^{-2} + 2\mu_2\alpha F_x] \tag{35}$$

and

$$B \equiv n_1\mu_2\alpha F_y\,. \tag{36}$$

We let the slow variation of $h_2$ and $k_2$ be given as

$$h_2 = e_2 \cos(Nt + D) \quad \text{and} \quad k_2 = e_2 \sin(Nt + D) \tag{37}$$

where $N$ is the outer satellite's constant precession rate, $e_2$ is its constant eccentricity and $D$ is a phase constant. The solution to eq. (34) is

$$h_1 = c \cos(At + \delta) + Kh_2 \quad \text{and} \quad k_1 = c \sin(At + \delta) + Kk_2 \tag{38}$$

where $c$ and $\delta$ are arbitrary constants of integration and $K \equiv B/(A-N)$. We can think of $h_1$ and $k_1$ as orthogonal components of an eccentricity vector $\vec{e}_1$ whose magnitude is $e_1$ and direction is $\tilde{\omega}_1$. Thus the solution (38) indicates that $\vec{e}_1$ moves in a circle about $K\vec{e}_2$ (as shown in Fig. 7) with an angular velocity $A$. If $|c| < |K|e_2$, $\phi$ librates. Otherwise it circulates through 360°. $K$ is a measure of the likelihood of libration in the sense that the larger its absolute value, the larger the area on the $h_1, k_1$ plane in which initial conditions will give $|c| < |K|e_2$. $Ke_2$ is the amplitude of the "forced" oscillation of $h_1$ and $k_1$; and $c$ is the amplitude of the free oscillation. The term "proper" eccentricity is often used to describe $c$. Note that as $\phi$ varies in libration or circulation, $e_1$ varies also.

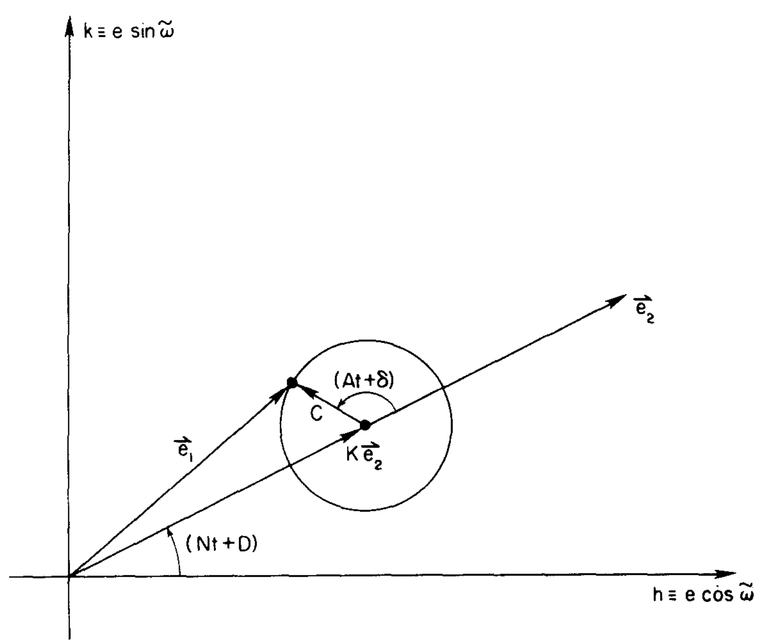


Fig. 7. Rhea's eccentricity vector, $\vec{e}_1$, with magnitude $e_1$ and direction $\tilde{\omega}_1$, is defined by a point which moves uniformly in a circle around $Ke_2$. Because $c < Ke_2$, $\tilde{\omega}_1$ librates about $\tilde{\omega}_2$.

Inserting numerical values for the Titan - Rhea case, $A = 10°$/yr, $B = 0.35°$/yr and $N = 0.5°$/yr. Therefore $K \simeq 0.035$. Observations of Rhea's motion give $c = 0.0003$ (Struve, 1928). $N$ is negligible compared to $A$. The value of $A$ is primarily due to Saturn's oblateness with only a slight correction from Titan's "ring" term. $B$ is directly proportional to $m_2$, so $K$ is approximately proportional to $m_2$. If Titan were not a giant satellite, that is if its mass were more typical of other Saturn satellites ($\sim 10^{-6}M$), Rhea's $e_1$ would have to be very small ($\sim Ke_2 \sim 10^{-5}$) for the libration to exist. It is unlikely that $e_1$ could remain so small in a real satellite system.

On the other hand, if $A$ and $N$ were nearly equal, $K$ could have a large value even if $m_2$ were small. $A$ would be smaller if $J$ were smaller or if $a_1$ were larger. $N$ would be larger if $m_2$

were influenced by the "ring" term of another satellite further out. Thus one could construct hypothetical satellite systems in which $A \simeq N$. $K$ would be enhanced by a 1 : 1 commensurability of these "natural" precession rates. (Here the "natural" precession rate is defined as the rate calculated from the "oblateness" and "ring" terms only.) The term "resonance" would be appropriate as the forced amplitude becomes large for commensurable natural frequencies.

In a sense, the Titan - Rhea interaction might be called a commensurability because, on the average, the apsidal precession periods are in a ratio of 1 : 1. However, it is more useful to reserve the term "commensurability" for cases in which the natural precession rates are equal and thus enhance the perturbations (Sinclair, private communication).

This analysis leads to a simple physical interpretation. We have already seen in Section 4 how an impulsive force exerted on a satellite in a direction radially outward from its primary will accelerate the nearest apse toward the longitude at which the force is exerted. In the absence of a commensurability of mean motions, the effect on Rhea can be modelled by a radial force exerted when Titan is at pericenter, the point at which the two orbits are closest together. Thus $\tilde{\omega}_1$ can be accelerated toward $\tilde{\omega}_2$. With suitable initial conditions $\tilde{\omega}_1$ will librate about $\tilde{\omega}_2$. If the natural precession rates, that is the rates independent of this effect, were 1 : 1 commensurable, Titan's mass could be much smaller and still maintain the libration.

The secular theory for a larger number of bodies can be developed in a manner similar to the Titan - Rhea case (e.g. Brouwer and Clemence, 1961a). A set of linear equations analogous to (34) is set up for each body in the system with terms on the right hand sides for each of the other bodies. An independent set of equations of the same form is considered for the behavior of inclinations and nodes. An eigenfunction solution is obtained. If any pairs of natural precession rates are 1 : 1 commensurable, the solution will have resonant oscillations of the relevant bodies' eccentricities or inclinations. A classical analysis of this type was performed by Brouwer and Van Woerkom (1950) for the planetary system. Their solution for the "secular" variations (which are in fact periodic in nature) of the planet's orbits have been confirmed by numerical integration of the orbits. Brouwer and Van Woerkom also found that particles with certain semi-major axes in the asteroid belt would experience secular resonance. Williams (1969) extended the secular theory of asteroid orbits to include higher inclinations and eccentricities. He found that the observed asteroid belt is remarkably depleted of particles near such orbits. Williams suggests that the oscillation of eccentricity and inclination in secular resonance may have carried asteroids into collision orbits with other planets. This mechanism provides a possible source for a substantial portion of the bodies which are known to have bombarded the terrestrial planets.

## 9. COUPLED LIBRATIONS

Several minor planets near mean motions commensurable with Jupiter's undergo strong variation due to secular terms in the disturbing function (Schubart, 1970). Numerical integration of the motions of asteroids near the 2 : 1 commensurability with Jupiter by Franklin *et al.* (1975) has revealed a previously undiscovered type of orbital behavior. Over a time scale of thousands of years, several asteroids are predicted to alternate between two modes of libration. In one mode the asteroid's perihelion longitude librates about that of Jupiter as in a secular resonance. In the other mode the asteroid's aphelion librates about the longitude of conjunction of the asteroid and Jupiter, as in a small-eccentricity resonance. The alternation between these two modes of libration suggests that they are coupled in some way. A highly simplified analysis keeps the problem solvable and the mechanism unobscured (Greenberg and Franklin, 1975).

Our model will have the ratio of orbital periods near 2 : 1, and nearly circular coplanar orbits with $m_2$ unperturbed by $m_1$. The relevant terms of the disturbing function at $m_1$ are

$$R = (Gm_2/a_2)[F_x(\alpha)e_1^{\,2} + F_y(\alpha)e_1 e_2 \cos\phi + F_A(\alpha)e_1\cos\sigma_A + F_B(\alpha)e_2\cos\sigma_B]\,. \qquad (39)$$

The arguments $\phi \equiv \tilde{\omega}_1 - \tilde{\omega}_2$ and $\sigma_A \equiv 2\lambda_2 - \lambda_1 - \tilde{\omega}_1$ are in essence the quantities that were found to librate alternately by Franklin *et al.* The argument $\sigma_B \equiv 2\lambda_2 - \lambda_1 - \tilde{\omega}_2$ is included because of its long period.

Lagrange's equations yield the following variation of orbital elements:

$$\dot{e}_1 = -\mu_2 n_1 \alpha[-F_y e_2 \sin\phi + F_A \sin\sigma] \qquad (40)$$

$$\dot{\tilde{\omega}}_1 = \frac{\mu_2 n_1 \alpha}{e_1}\,[2F_x e_1 + F_y e_2\cos\phi + F_A\cos\sigma] \qquad (41)$$

$$\dot{n}_1 = -3\mu_2 n_1^{\,2}\alpha[F_A e_1 \sin\sigma + F_B e_2 \sin(\sigma + \phi)] \qquad (42)$$

where $\sigma \equiv \sigma_A$ and $\sigma_B$ has been replaced by $\sigma + \phi$. $\varepsilon_1$ is nearly constant. The variation of the critical arguments is given by

$$\dot{\phi} = \dot{\tilde{\omega}}_1 \qquad (43)$$

and

$$\dot{\sigma} = 2n_2 - n_1 - \dot{\tilde{\omega}}_1\,. \qquad (44)$$

After inserting (41) into (43) and (44), we will have a set of equations, (40), (42), (43) and (44), whose solution will describe the behavior of $\phi$ and $\sigma$.

In order to simplify these equations and permit an analytic solution, we note that changes in $n_1$ are of order $e$. Thus any changes in $n_1$ on the right sides of (40), (43) or (44) will be assumed negligible. This reduces the problem to three first-order differential equations. Integration of (44) yields $\sigma = At - \phi$ where $A \equiv 2n_2 - n_1$ and where $t \equiv 0$ when $\sigma = -\phi$. This expression can be substituted into (40) and (43), a pair of equations which can be linearized by substitution of the elements $h \equiv e_1\cos\phi$ and $k \equiv e_1 \sin\phi$ to obtain

$$\dot{h} = -\mu_2 \alpha n_1 [F_x h + F_A \sin At] \qquad (45)$$

$$\dot{k} = \mu_2 \alpha n_1 [F_x k + F_y e_2 + F_A \cos At]\,. \qquad (46)$$

The solution of these equations is

$$h = C\cos(2\mu_2\alpha n_1 F_x t + \delta) + \frac{F_A \cos At}{A/(\mu_2\alpha n_1) - 2F_x} - \frac{F_y e_2}{2F_x} \qquad (47)$$

$$k = C\sin(2\mu_2\alpha n_1 F_x t + \delta) + \frac{F_A \sin At}{A/(\mu_2\alpha n_1) - 2F_x} \qquad (48)$$

where $C$ and $\delta$ are arbitrary constants of integration. This solution can be represented in $(h,k)$ rectangular coordinates, equivalent to $(e,\phi)$ polar coordinates, as the sum of three vectors: (i) $h = -F_y e_2/(2F_x)$, $k = 0$, (ii) a vector of fixed magnitude $C$ circulating at rate $2\mu_2\alpha n_1 F_x$ and (iii) a vector of fixed magnitude $F_A/(A/(\mu_2\alpha n_1) - 2F_x)$ circulating at rate $A$.

In order to interpret this result, we require some numerical values. The semi-major axis of 9594 P-L, a typical asteroid exhibiting the coupled librations remains nearly constant and corresponds to the value $\alpha \doteq 0.643$, so that $F_x = 0.426$, $F_y = -0.644$, $F_A = -1.262$ and $2(n_2/n_1) - 1 = 0.032$. For Jupiter, $m_2/M \cong 10^{-3}$ and $e_2 \cong 0.05$. Using these values as sufficiently accurate, we obtain

$$h = C \cos(6.9 \times 10^{-3}t + \delta) - 0.026 \sin(0.40t) + 0.038 \tag{49}$$

$$k = C \sin(6.9 \times 10^{-3}t + \delta) - 0.026 \sin(0.40t) \tag{50}$$

where $t$ is expressed in units of Jupiter's orbital period.

In polar coordinates, $(e_1,\phi)$ describes a circle of radius 0.026 about a point $P$. $P$ in turn, as is shown in Fig. 8, slowly circulates about point $h = 0.038$, $k = 0$. If $(0.038 - 0.026) < C < (0.038 + 0.026)$, $\phi$ will alternately librate and circulate. During the libration the mean value of $\phi$ will gradually increase. At each transition between libration and circulation and vice versa, the value of $e_1$ momentarily drops nearly to zero. Equations (49) and (50) imply that when $\phi$ circulates (i.e. when $(6.9 \times 10^{-3}t + \delta) \cong \pi$, as in Fig. 8 (E)), $\phi$ itself is approximately equal to $0.4\,t + \pi$. Since $\sigma = 0.4t - \phi$, it then follows that $\sigma$ librates about $\pi$. On the other hand, when $\phi$ librates as in Fig. 8 (A), $\sigma$ must circulate.

The relatively short-period variation of $\phi$ is forced by the $\sigma$ term in the disturbing function. If $n_2/n_1$ were not close to $\frac{1}{2}$, $|A|$ would be much larger and the behavior of $\phi$ would be either perpetual circulation or libration, depending on the value of $C$. (The Rhea - Titan system is an example of the latter case.) In either situation terms containing $\sigma$ could then be neglected from the disturbing function without substantially changing the results—just as we have left out many other terms from $R$ in eq. (39). On the other hand, even with $n_2/n_1 \approx \frac{1}{2}$, $\phi$ can circulate or librate perpetually given a suitable initial value of $C$. However, as shown above, in such a case the period is controlled by the $\sigma$ term.

The validity of the various assumptions in this analysis has been discussed by Greenberg and Franklin (1975). The properties revealed by this analysis are all essential elements of the behavior discovered by Franklin *et al.* (1975). The period of libration or circulation of $\phi$ and $\sigma$ is given by $2\pi/0.4$, which is ~200 yr, in good agreement with their result. The period of alternation between libration and circulation is $2\pi/6.9 \times 10^{-3}$ or ~11,500 yr, which for 9594 P-L is about three times their value. J. G. Williams (private communication) has noted that this discrepancy can be accounted for by incorporating the variation in $\lambda_1$ due to first order variation in $n_1$ into eqs. (40) and (41). No qualitative changes are introduced into the solution, but this correction yields a long period of alternation between libration and circulation ~4000 yr, in agreement with the numerical evaluation of Franklin *et al.*

Physically, we may interpret the coupled behavior by noting that the apsides are accelerated toward the longitude of conjunction of the asteroid and Jupiter by the small eccentricity mechanism described in Section 4. They are also accelerated toward Jupiter's perihelion longitude by the secular resonance mechanism. The coupled resonance described here is simply the combined response to these two perturbing effects.

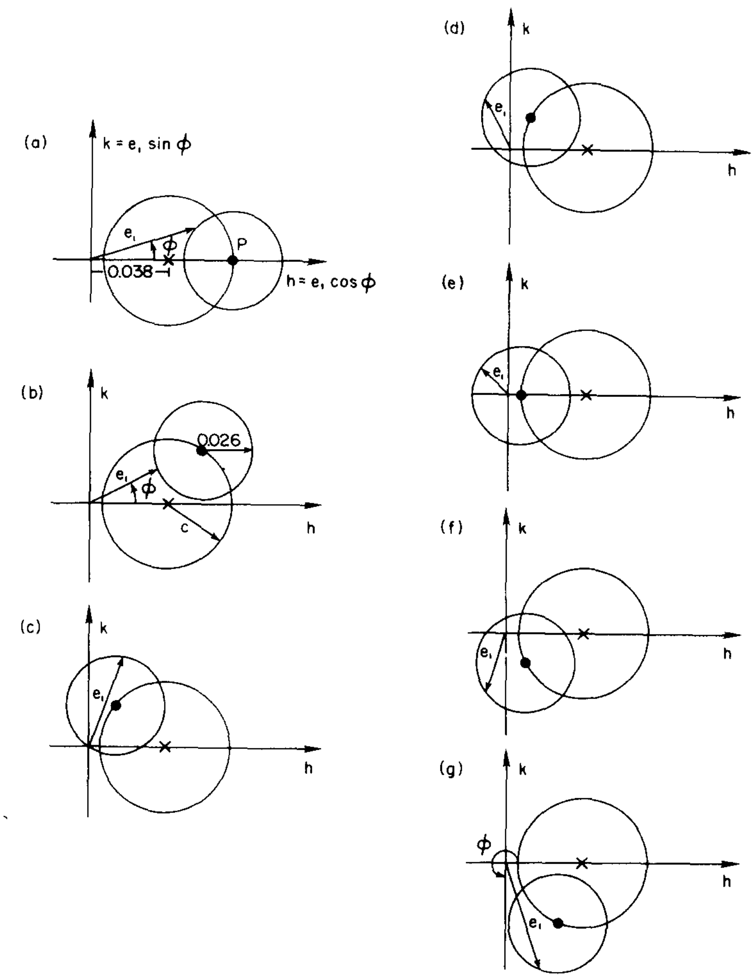


Fig. 8. The $h$, $k$ vector behaves according to eq. (49) and (50). The point $(h,k)$ rotates counterclockwise about a circle of radius 0.026, whose center, $P$, in turn rotates relatively slowly in a counterclockwise direction about a circle of radius $c$ centered at $h = 0.038$, $k = 0$. $\phi$ librates in (a), (b), (c), and (g) and circulates otherwise. (From Greenberg and Franklin, 1975).

## 10. THE LAPLACE RELATION

Three Galilean satellites of Jupiter, Io (1), Europa (2) and Ganymede (3), are involved in another type of coupled libration, the stability of which was first demonstrated by Laplace. The quantity $\Theta \equiv \lambda_1 - 3\lambda_2 + 2\lambda_3$ is locked to a value of 180° with a libration amplitude $\lesssim 0\overset{\circ}{.}03$, too small to detect (DeSitter, 1931). This expression implies that whenever Europa and Ganymede are in conjunction with respect to Jupiter (i.e. whenever $\lambda_2 = \lambda_3$), Io is 180° away (Fig. 9). The three satellites are prevented from lining up on the same side of Jupiter. The commensurability relation between mean motions is given by differentiating the librating quantity to obtain $n_1 - 3n_2 + 2n_3 \simeq 0$. Moreover, taken by adjacent pairs the mean motions have ratios of 2 : 1. Thus, expressing the resonance variable as $(2\lambda_3 - \lambda_2) - (2\lambda_2 - \lambda_1)$ demonstrates that the longitudes of conjunction of pairs are separated by 180°.

To analyze this resonance we follow the theory of Souillart (Tisserand, 1896) and note that the important terms are those whose arguments are slowly varying due to the 2 : 1 commensur-

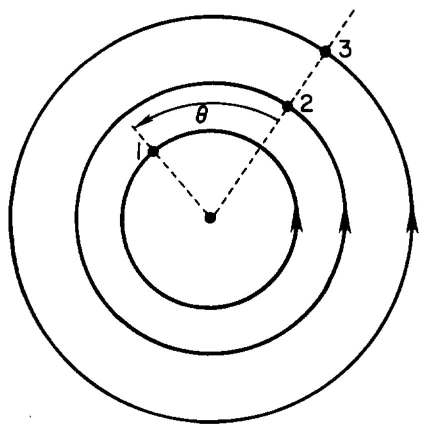


Fig. 9. $\Theta$ is the longitude of satellite 1 relative to satellite 2 at the instant of conjunction of satellites 2 and 3. For Io, Europa and Ganymede $\Theta = 180^\circ$. (From Greenberg, 1975).

abilities between adjacent satellites. Retaining only such terms, and these terms only to lowest order in eccentricity, yields—for example—the disturbing function at satellite 1:

$$R_1 = (Gm_2/a_2)\; F_A e_1 \cos(2\lambda_2 - \lambda_1 - \tilde{\omega}_1) + F_B e_2 \cos(2\lambda_2 - \lambda_1 - \tilde{\omega}_1) \qquad (51)$$

which is identical to $R$ for the Enceladus - Dione case (eq. (13)).As the eccentricities are small and variable, we redefine $h \equiv e \cos\tilde{\omega}$ and $k \equiv e \sin\tilde{\omega}$. Also let $u_{ij} = 2\lambda_j - \lambda_i$. Thus

$$R_1 = (Gm_2/a_2)[F_A(h_1 \cos u_{12} + k_1 \sin u_{12}) + F_B(h_2 \cos u_{12} + k_2 \sin u_{12})] \qquad (52)$$

with similar expressions for $R_2$ and $R_3$.

In order to investigate the behavior of $\Theta$, we must evaluate

$$\ddot{\Theta} = \dot{n}_1 - 3\dot{n}_2 + 2\dot{n}_3. \qquad (53)$$

(Variation of $\varepsilon$, the mean longitude at epoch, is negligible.) In order to demonstrate evaluation of expression (53), let us consider in detail the first term only. Lagrange's planetary equations give

$$\dot{n}_1 = -3\mu_2 n_1{}^2 \alpha [F_A(h_1 \sin u_{12} - k_1 \cos u_{12}) + F_B(h_2 \sin u_{12} - k_2 \cos u_{12})]. \qquad (54)$$

The most important part of this variation is not the first order (in $\mu$) solution obtained by considering the elements on theright side of (54), except for $u_{12}$, to be constant. The second order effect due to variation of $h$'s and $k$'s on the right side of (54) is more significant for two reasons: (i) it is not proportional to the small eccentricities and (ii) it contains the fixed argument $\Theta$ and the small divisor $n_1 - 2n_2$. To demonstrate these second order effects consider the variation of $h_1$ given by eq. (33):

$$\dot{h}_1 = -\mu_2 n_1 \alpha F_A \sin u_{12}. \qquad (55)$$

Integration yields the first order variation

$$\Delta h = \frac{\mu_2 n_1 \alpha F_A \cos u_{12}}{(2n_2 - n_1)}. \qquad (56)$$

Similar results are obtained for the remaining $h$'s and $k$'s.

Thus the important second order variation of $n_1$ is given by

$$\dot{n}_1 = \frac{\partial \dot{n}_1}{\partial h_1} \Delta h_1 + \frac{\partial \dot{n}_1}{\partial k_1} \Delta k_1 + \frac{\partial \dot{n}_1}{\partial h_2} \Delta h_2 + \frac{\partial \dot{n}_1}{\partial k_2} \Delta k_2 \ . \tag{57}$$

The appropriate substitutions and algebraic and trigonometric manipulation yield

$$\dot{n}_1 = \frac{3\mu_2\mu_3\alpha^2 F_A F_B n_1{}^2 n_2}{2n_2 - n_1} \sin\Theta \ . \tag{58}$$

Similar evaluation of $\dot{n}_2$ and $\dot{n}_3$ and substitution into eq. (53) yields

$$\ddot{\Theta} = \left[\frac{\mu_2\mu_3}{a_1{}^2} + \frac{9\mu_1\mu_2}{a_2{}^2} + \frac{4\mu_1\mu_2}{a_3{}^2}\right]\left[\frac{3}{8}\,\frac{a_1 a_2 n_1{}^2 F_A F_B}{(2n_2 - n_1)}\right] \sin\Theta \ . \tag{59}$$

This equation for the behavior of $\Theta$ is analogous to that of a pendulum. Evaluation of the coefficient shows it to be positive. Thus $\Theta$ is stable at the value 180°. This mechanism can also be described qualitatively in the following manner (this description is necessarily simplified to include only those effects which contribute significantly to the behavior of $\Theta$).

First consider the effect of Ganymede (3) on Europa (2). If the ratio of their orbital periods were exactly 2 : 1, the longitude of conjunction of this pair would be fixed. In fact, because the ratio is only approximately 2 : 1, conjunction regresses at a rate of about 0°.74/day. As a result of the planet's oblateness and the secular effects of the other satellites, Europa's pericenter advances. Thus conjunction circulates in the retrograde direction relative to the apsides of Europa's orbit.

We can approximate the effect of Ganymede on Europa as an impulsive force exerted radially outward from Jupiter whenever conjunction occurs. We know that such a radial force tends to cause a regression of the apsides if exerted near pericenter and an advance of apsides if exerted near apocenter. As conjunction passes Europa's pericenter the advance of the apsides slows down; as conjunction passes Europa's apocenter, the advance of apsides speeds up. Thus Europa's pericenter spends more time near the longitude of conjunction than does the apocenter.

Next consider the effect of Europa (2) on Io (1). Suppose conjunction of these two satellites occurs after Europa's pericenter, but before apocenter. Since the velocities are diverging at such a conjunction, the two satellites are closest to one another shortly before conjunction. Thus when the satellites are closest (and the perturbing force is greatest), Europa is slightly ahead of Io. Therefore Io gains orbital energy and its period increases. As a result, conjunction is accelerated forward towards Europa's apocenter. Similarly, if conjunction occurs after apocenter and before pericenter, conjunction is accelerated back towards Europa's apocenter. This effect is rather weak due to the small eccentricity. Therefore it operates over such a long time scale that Europa's apocenter may be considered to be, on the average, 180° from conjunction of Europa and Ganymede. Thus conjunction of Io and Europa tends to be restored to a longitude 180° from conjunction of Europa and Ganymede.

Similar consideration of the effects of each of the three satellites on each of the others supports the same conclusion: $\Theta$ is stable at 180°. Sinclair (1975) noted that, in addition to $\Theta$, the quantities $2\lambda_2 - \lambda_1 - \tilde{\omega}_1$, $2\lambda_2 - \lambda_1 - \tilde{\omega}_2$ and $2\lambda_3 - \lambda_2 - \tilde{\omega}_2$ also librate.

It is clear that the mechanism governing the Laplace relation among the Galilean satellites depends on pairwise commensurabilities. However, there is another Laplace relation in the solar system which does not include pairwise commensurabilities: the interaction of Miranda, Ariel and Umbriel, the three inner satellites of Uranus, which we shall refer to as 1, 2 and 3, respectively. (This notation is in violation of the classical numbering sequence.) In this case $\Theta$ circulates slowly ($\dot{\Theta} = -0\overset{\circ}{.}08$/day). But theory shows that if the satellites' masses were slightly larger or if $\Theta$ were slightly closer to zero, $\Theta$ would librate (Greenberg, 1976).

In the Uranus system the quantity $|n_1 - 2n_2|$ is not much smaller than the mean motions. Thus the coefficient in eq. (59) is not enhanced by a small divisor. More fundamentally, the terms retained in the disturbing function (eq. (52)) are not the only important terms for the Uranus satellites. Nevertheless, the system is simplified in some respects. Photometric estimates indicate that the mass of Miranda is small enough for us to ignore its effects on the other satellites. The lack of pairwise commensurabilities ensures that no first-order (in $\mu$) effects can be important, because, aside from secular terms (Greenberg, 1975), there are no long-period terms in the first-order variation. For our purposes here, the orbits will be considered coplanar.

Sinclair (1975) notes that terms of *zeroth* order in eccentricity can be as significant in the Uranus system as terms considered for the Galilean satellites. For example, one term in $\dot{n}_1$ due to satellite 2 is

$$\dot{n}_1 = \mu_2 A \sin(\lambda_1 - \lambda_2) \tag{60}$$

where $A$ is a function of $n_1$ and $n_2$. The second order variation of $n_1$ is due in part to variation of $n_2$ on the right-hand side of (60). One term in $\dot{n}_2$ due to satellite 3 is

$$\dot{n}_2 = \mu_3 B \sin 2(\lambda_2 - \lambda_3). \tag{61}$$

Integrating (61) yields, to first order in $\mu$,

$$\Delta n_2 = -\frac{\mu_3 B}{n_2 - n_3} \cos 2(\lambda_2 - \lambda_3).$$

Part of the second order variation of $dn_1/dt$ is thus given by

$$\dot{n}_1 = \mu_2(\partial A/\partial n_2)\Delta n_2 \sin(\lambda_1 - \lambda_2)$$

$$= -\mu_2\mu_3(\partial A/\partial n_2)\frac{B}{(n_2 - n_3)} \cos 2(\lambda_2 - \lambda_3)\sin(\lambda_1 - \lambda_2)$$

$$= -\tfrac{1}{2}\mu_2\mu_3(\partial A/\partial n_2)\frac{B}{(n_2 - n_3)} \sin\Theta + \text{other terms.} \tag{62}$$

In this way the argument $\Theta$ can appear in the expression for $\ddot{\Theta}$ (eq. (53)) as a combination of arguments such as $\lambda_1 - \lambda_2$ and $2(\lambda_2 - \lambda_3)$ from the *zeroth* order (in $e$) portion of the disturbing functions. Moreover, as Sinclair has noted, combinations of argument $3(\lambda_1 - \lambda_2)$ with $2(\lambda_1 - \lambda_3)$ and of argument $\lambda_1 - \lambda_3$ with $3(\lambda_2 - \lambda_3)$ also yield second order (in $\mu$) perturbations with long-period argument $\Theta$. Combinations of multiples of these arguments yield terms with multiples of $\Theta$ as arguments. No terms in $\ddot{\Theta}$ contain small divisors. Thus, in this case, there is no reason for these additional terms to be much smaller than the term which dominated in the Jovian system.

In most resonance cases, long-period variations in orbital elements depend on a system's asymmetry due to significant eccentricity or inclination. If eccentricities and inclinations are zero, symmetry would seem to prevent any long-period variation of mean motions. How can Sinclair's mechanism, which is independent of eccentricity and inclination, yield such long-period effects? With three satellites involved the symmetry does not carry through to second order in $\mu$. For example, consider the direct effect of Umbriel (3) on Ariel (2). Just before conjunction of these two satellites, Ariel will gain energy. Just after conjunction, Ariel will lose energy. Thus Ariel will be moving relatively slowly at conjunction with Umbriel. Now let us consider the effect of this disturbed satellite on Miranda (1). Suppose $\Theta$ is a few degrees less than zero. At conjunction of Ariel and Umbriel, Miranda is a few degrees behind them. We have already seen that Ariel is moving more slowly than usual at this point. Thus as viewed from Miranda, Ariel spends less time ahead of Miranda than behind. Thus Miranda suffers a net loss of energy and speeds up, yielding $\ddot{\Theta} > 0$. Similarly, if $\Theta > 0$, $\ddot{\Theta} < 0$. Thus this simplified description indicates that $\Theta$ would be stable at 0.

Evaluation of all significant terms in the expression for $\Theta$ (Greenberg, 1975; 1976) reveals that terms generated by the Souillart analysis are comparable to Sinclair's zero eccentricity effects. A generalized Souillart approach must be used which includes all terms in R of first order in eccentricity, not just the 2 : 1 terms. It is found that the potential well which is centered at $\Theta = 0$ has a small bump at the bottom, so that $\Theta$ would actually be stable at $\pm 36^\circ$. In fact, $\Theta$ is circulating through this potential topography, albeit slowly. The circulation is slow enough for observable variation in Miranda's longitude to be produced. Given the scarcity of observations of Miranda, we can now only use this theory to place an upper limit on $\mu_2\mu_3$, the product of the masses of Ariel and Umbriel, of $\sim 10^{-9}$ (Greenberg, 1976).

## 11. TROJAN ASTEROID ORBITS

The Trojan asteroids are locked to longitudes $60^\circ$ from that of Jupiter. It follows that their mean motions are in 1 : 1 resonance with Jupiter's. Since it also follows that their semi-major axes oscillate about the value of Jupiter's distance from the sun, our expansion of the disturbing function becomes useless in this case.

A convenient way to handle this case analytically is to assume Jupiter's orbit to be circular and then to make use of the well-known restricted three-body problem (e.g. Danby, 1962). The motion of an asteroid (which is treated as an infinitesimal test particle) can then be studied in a reference frame rotating with Jupiter's mean motion about the Sun - Jupiter center of mass. In this rotating frame, the motion is governed by the gravitational potential, by the centrifugal force and by the Coriolis force. The centrifugal force depends on position only, so it can be incorporated with the gravitational potential into a corrected potential field. In a search for equilibrium points a particle's velocity can be set equal to zero. A particle will be in equilibrium at any point at which the gradient of the corrected potential is zero. The locations about which the observed Trojans oscillate (generally called the $L_4$ and $L_5$ points) are indeed such equilibrium points. However, an interesting fact, often glossed over in treatments of the restricted problem, is that these points represent maxima, not minima, of the corrected potential. How can these be stable equilibria? The answer is that as soon as a particle starts moving near these points, the velocity-dependent Coriolis force plays a dominant role which tends to restore the stable configuration.

This configuration shares the characteristics, common to many resonances, that the gravitational perturbations act to prevent close approaches of bodies to the perturber.

## 12. HIGHER ORDER COMMENSURABILITIES

With the exceptions of the 1 : 1 resonance and secular resonances, the commensurabilities discussed so far have involved ratios of periods of the form $(j + 1) : j$ where $j$ is an integer. In general, the mechanisms are dominated by the repetitious forces exerted near conjunction, where the separation of interacting bodies is minimal. An outstanding exception is the Neptune - Pluto case, in which Pluto's large eccentricity introduces additional minima that dominate the interaction. The largest value of $j$ for a known $(j + 1) : j$ resonance is 3. For higher values of $j$, semi-major axes would be nearly equal so that even moderate eccentricities would lead to the sort of complications found in the Neptune - Pluto case.

For higher order commensurabilities of the form $(j + q) : j$, resonance effects are considerably weakened for integer $q > 1$. Conjunctions occur at $q$ different longitudes. For example, in a 3 : 1 commensurability with small eccentricities the inner body would make $1\frac{1}{2}$ revolutions to the outer body's $\frac{1}{2}$ revolution between conjunctions. Thus any effects at one conjunction would be largely neutralized by effects at the next conjunction. The d'Alembert rules reflect this weakened mechanism by reducing coefficients of the relevant terms in the disturbing function. In the 3 : 1 case the argument $\Theta \equiv 3\lambda_2 - \lambda_1 - 2\tilde{\omega}_1$ would have a long period but it is clearly of second order in $e_1$ according to the d'Alembert rules. Note that the longitude of conjunction in a 3 : 1 resonance would be at $(3\lambda_2 - \lambda_1)/2$, so $\Theta$ would represent twice the longitude of conjunction measured from $\tilde{\omega}_1$. If $\Theta$ librates about a fixed value $\alpha$, then conjunction occurs at the two longitudes $\alpha$ and $\alpha + 180°$.

Resonances with $q > 1$ are so weakened in this way that they only exist when the eccentricities are large enough for interactions well-removed from conjunction to dominate, as in the Neptune - Pluto case. Consider the example of asteroid Alinda (887) with eccentricity $\approx 0.54$ (Marsden, 1970; Janiczek *et al.*, 1972). This asteroid is locked to a 3 : 1 resonance with Jupiter. Reference to Fig. 10 indicates that the effect of Jupiter on Alinda at aphelion tends to maintain the resonance by a means analogous to Cohen and Hubbard's Neptune - Pluto mechanism. $\Theta$ librates about a value near 180°. This might seem to imply that conjunction occurs 90° before and after perihelion. However, for such a large eccentricity we must be careful to note that Alinda's mean longitude would be very different from its true longitude, so each conjunction actually occurs about 65° before and after perihelion.

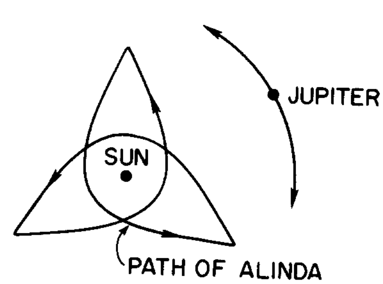


Fig. 10. Approximate path of asteroid Alinda (887) in a frame rotating with Jupiter's motion. Libration is shown by oscillation of Jupiter's position. The effect of Jupiter on Alinda prevents conjunction near the latter's aphelion. The mechanism is analogous to the Neptune - Pluto interaction, Section 7.

Other examples of high-order asteroid resonances are discussed by Janiczek *et al.* (1972), Ip and Mehra (1973) and Marsden (1970). Two of the most remarkable examples involve the asteroids

Toro (1685) and Ivar (1627). Toro apparently alternates between libration in a 13 : 5 resonance with Venus and libration in an 8 : 5 resonance with the Earth. Ivar undergoes libration in an 11 : 28 resonance with the Earth. Despite the extremely high order, this libration is apparently quite stable. As in the Neptune - Pluto case, the analytical obstacles in these large-eccentricity cases are overwhelming. They are generally discovered and investigated by numerical techniques.

## 13. CONCLUSION

Orbit - orbit resonances occur in a wide variety of types: small eccentricity (Enceladus - Dione), larger eccentricity (Titan - Hyperion), very large eccentricity (Neptune - Pluto), inclination (Mimas - Tethys), three satellite (Io - Europa - Ganymede) and secular. In this paper I have discussed various examples to illustrate the variety and underlying similarities of these phenomena. Because of this specific purpose, a great many other examples have been ignored as they cast no additional light on the resonance mechanism. In particular the asteroid belt, which spans a region of low order commensurabilities with Jupiter, contains numerous examples of the various types. Recent reviews by Peale (1976) and Hartmann *et al.* (1976) provide information on these cases. Older, but still useful compendia of resonance data are by Brouwer and Clemence (1961b) and Hagihara (1972).

The close correspondence between the terms of the expanded disturbing function and the types of resonance should now be clear. Most interactions can be represented by the Fourier decomposition of the mutual forces into a d'Alembert series. When orbits have commensurable periods repetitive configurations tend to enhance their mutual effects. Correspondingly, certain terms in the expansion have long periods. Actually, for any pair of planets or satellites, an integer ratio can be selected which is arbitrarily close to the ratio of mean motions. Hence there are always long-period terms. However, unless the mean motions are commensurable, the integer ratio will be of high order. The d'Alembert rules as well as physical intuition assure us that the long-period terms are only important in cases of small-integer commensurability.

Long-period effects can result in enhanced perturbations which may tend to maintain the periodicity of a system. The expanded disturbing function may thus serve as a catalog of stable periodic orbits which would be possible for properly chosen initial conditions. Conditions in the solar system have been such that it contains a remarkably diverse collection of resonances.

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Table 1. Properties of some orbital interactions discussed in the text

| Planet or Satellite | Sidereal Period | $i$ | $e$ | $\mu$ | Librating Argument | Mean Value | Libration Amplitude | Libration Period | Reference |
|---|---|---|---|---|---|---|---|---|---|
| Rhea | 4.52d | $0^{\circ}.3$ | ~0.001 | $\sim 4\times10^{-6}$ | | | | | |
| Titan | 15.95d | $0^{\circ}.3$ | 0.03 | $2.4\times10^{-5}$ | $\tilde{\omega}_T-\tilde{\omega}_R$ | 0 | $9^{\circ}.5$ | 38yr | Struve 1928 |
| Hyperion | 21.28d | $0^{\circ}.5$ | 0.10 | $\sim 2\times10^{-7}$ | $4\lambda_H-3\lambda_T-\tilde{\omega}_H$ | 180° | 36° | 1.75yr | Woltjer 1928 |
| Enceladus | 1.370d | $0^{\circ}.0$ | 0.0044 | $\sim 1\times10^{-7}$ | | | | | |
| Dione | 2.737d | $0^{\circ}.0$ | 0.0022 | $2\times10^{-6}$ | $2\lambda_D-\lambda_E-\tilde{\omega}_E$ | 0 | <1° | 12yr | W. H. Jefferys (private communication) |
| Mimas | 0.942d | $1^{\circ}.5$ | 0.02 | $6\times10^{-8}$ | | | | | |
| Tethys | 1.888d | $1^{\circ}.1$ | 0.00 | $1\times10^{-6}$ | $4\lambda_T-2\lambda_M-\Omega_T-\Omega_M$ | 0 | 97° | 71yr | Allan 1969 |
| Neptune | 164.8yr | $1^{\circ}.8$ | 0.01 | $5\times10^{-5}$ | | | | | |
| Pluto | 248.4yr | $17^{\circ}.2$ | 0.25 | $\sim 3\times10^{-7}$ | $3\lambda_P-2\lambda_N-\tilde{\omega}_P$ | 180° | 76° | $2\times10^{4}$yr | Cohen and Hubbard 1965 |
| Io | 1.769d | $0^{\circ}.0$ | 0.00 | $4.7\times10^{-5}$ | | | | | |
| Europa | 3.551d | $0^{\circ}.5$ | 0.00 | $2.6\times10^{-5}$ | $\lambda_I-3\lambda_E+2\lambda_G$ | 180° | $\lesssim 0^{\circ}.03$ | ~6yr | DeSitter 1931 |
| Ganymede | 7.156d | $0^{\circ}.2$ | 0.00 | $7.8\times10^{-5}$ | | | | | |
| Miranda | 1.413d | ? | ~0.01 | $\sim 3\times10^{-6}$ | | | | | |
| Ariel | 2.520d | 0 | ~0.003 | $\sim 6\times10^{-5}$ | $\lambda_M-3\lambda_A+2\lambda_U$ (circulates) | - | - | 12 yr | Greenberg 1976 |
| Umbriel | 4.144d | 0 | ~0.001 | $\sim 2\times10^{-5}$ | | | | | |
| Jupiter | 11.86yr | $1^{\circ}.3$ | 0.048 | $9.6\times10^{-4}$ | | | | | |
| Alinda | 3.99yr | $9^{\circ}.1$ | 0.54 | $\sim 10^{-16}$ | $3\lambda_J-\lambda_A-2\tilde{\omega}_A$ | 180° ? | 113° | 360yr | Marsden 1970 |

## Appendix
## Definitions of Orbital Elements

In order to summarize the celestial mechanical terminology and define the notation used in the text, the following definitions are provided. For a more detailed discussion the reader should use a reference such as Danby (1962).

The figure of a satellite's orbit can be specified by its eccentricity, $e$, which determines the orbit's shape, and its semi-major axis, $a$, which determines its size. The plane of the orbit is usually specified by its inclination, $i$, to the inertial reference plane and the longitude of the ascending node, $\Omega$, i.e. the angular position on the reference plane of the point in the orbit where the satellite ascends across the reference plane. The orientation of the ellipse on its own plane is given by the argument of pericenter, $\omega$, the angle between the line of the ascending node and the line from focus to pericenter. The longitude of pericenter, $\tilde{\omega} \equiv \omega + \Omega$, can be used instead of $\omega$. It is particularly useful when $i = 0$ and the nodes are not well-defined. The position on the orbit can be given by the true anomaly, $\eta$, the angular position measured from pericenter in the direction of motion. Since real satellites move in continually perturbed orbits, these orbital elements refer only to the instantaneously "osculating" orbit which would ensue if the perturbing forces could be suddenly eliminated.

The mean motion, $n$, the time-averaged angular velocity on the orbit, is related to the semi-major axis in unperturbed elliptical motion by Kepler's third law, $n^2a^3 = G(M + m)$, where G is the gravitational constant and $M$ and $m$ are the central body and satellite masses respectively. Thus $n$ can be used as an alternative to $a$ as an orbital element.

The mean anomaly, $M$, defined as the product of the mean motion and the time after pericenter passage, can be used instead of $\eta$ to specify position on the orbit. Another alternative is to use the mean longitude, $\lambda = \tilde{\omega} + M$. At a given time, the position on the orbit can also be specified by the mean longitude at epoch, $\varepsilon$, defined by

$$\lambda = \int_o^t n dt + \varepsilon \ .$$

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