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AN EXPLANATION OF THE FREQUENT OCCURRENCE OF COMMENSURABLE MEAN MOTIONS IN THE SOLAR SYSTEM

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Summary

In this paper a tidal origin of commensurable satellite mean motions is proposed. It is shown that special cases of commensurate mean motions are not disrupted by tidal forces. Furthermore, at least four, and probably seven, of the best examples of commensurabilities have this stability. The significance of these stable configurations to the evolution of satellite systems is discussed and some inferences are drawn about tidal dissipation in Jupiter and Saturn.

1. *Introduction.*—The existence of near-commensurabilities among the mean motions of the satellites and planets has been known for many years. The most famous of these commensurabilities involves the Jovian satellites Io, Europa and Ganymede. Within observational accuracy (9 significant figures) the mean motions (n_1, n_2 and n_3 respectively) of these satellites obey the relation $n_1 - 3n_2 + 2n_3 = 0$. The motions of these satellites have been studied in great detail, first by Laplace, and subsequently by many other authors. In addition to this three-body commensurability, several cases of near-commensurabilities between the mean motions of two satellites have also been known for quite some time. The motions of these pairs of satellites have also been intensively studied since they yield data from which a determination of the satellite masses can be made. More recently, A. E. Roy and M. W. Ovenden (**1, 2**) have examined the mean motions of pairs of planets and satellites in a new light. They considered the question of whether the observed number of near-commensurate pairs of mean motions in the solar system was too great to have arisen from a random distribution of mean motions. As this paper is intended to provide answers to several intriguing questions that they raised, we shall begin with a discussion of the contents of their two papers.

In their first paper, the authors arived at the conclusion that the preference for near-commensurate mean motions in the solar system is inconsistent with the assumption of a random distribution of mean motions for the planets and satellites. A sketch of their proof of this important result will be presented next.

Before we can prove anything, a sharper definition of near-commensurate mean motions must be given. Let n_1 and n_2 ($n_1 > n_2$) be the mean motions of two bodies about a common centre of force. If two integers, A_1 and A_2, exist such that

$$\left|\frac{n_2}{n_1} - \frac{A_2}{A_1}\right| = \epsilon$$

where ϵ is a small positive number, then these mean motions are said to be nearly commensurate in the ratio A_2/A_1. Since the ratio n_2/n_1 can always be approximated with arbitrary accuracy by the ratio of two integers, it is necessary to limit the size of the integers considered. In their paper, Roy and Ovenden arbitrarily set this limit for A_1 at seven. This restriction to small integers in no way limits the scope of their discussion. In fact, it can easily be shown from classical perturbation theory that the dynamical consequences of near-commensurabilities decrease as A_1-A_2 increases. Using our definition of near-commensurability, we may assign two integers, A_1 and A_2, to every pair of mean motions, n_1 and n_2, whose ratio $n_2/n_1 \geq 1/7$. Since the smallest difference between adjacent fractions is $1/6-1/7=1/42$ there can be at most one pair of integers, A_1, A_2, for each pair of mean motions, n_1, n_2, such that

$$\left|\frac{n_2}{n_1}-\frac{A_2}{A_1}\right| = \epsilon \leq 1/82 = 0{\cdot}01190.$$

From A_1 and A_2, with $A_1 \leq 7$, we can form 17 fractions with values between 1/7 and 1. Thus, given $\epsilon_0 \leq 0{\cdot}01190$ the probability that a randomly chosen ratio in the range 1/7 to 1 lies within ϵ_0 of one of these fractions is

$$P_{\epsilon_0} = 17 \times 2\epsilon_0 \times 7/6 = 39{\cdot}67\,\epsilon_0.$$

Roy and Ovenden considered 46 pairs of mean motions, and compiled a table which compares $46\,P_{\epsilon_0}$, for various $\epsilon_0 \leq 0{\cdot}01190$, with the observed number of pairs of mean motions for which a near-commensurability exists with $\epsilon \leq \epsilon_0$. This table, minus their control distribution data, is reproduced below.

TABLE I

ϵ_0	0·0119	0·0089	0·0059	0·0030	0·0015
$46P_{\epsilon_0}$	21·7	16·2	10·8	5·3	2·5
Observed number of pairs of mean motions with $\epsilon \leqslant \epsilon_0$	33	26	20	12	6

As Roy and Ovenden pointed out, there are two reasons why the observed number of near-commensurabilities, which are listed in their table, might be misleading. In the first place, if n_2 is nearly commensurable with both n_1 and n_3, then it may also be the case that n_3 is nearly commensurate with n_1. If this is so, then it is unclear whether the commensurability between n_1 and n_3 should be considered as an independent one. In their first paper, Roy and Ovenden showed that this problem of "multiple counts" was likely to affect the number of independent observed commensurabilities listed in Table I, by 2 or 3 for $\epsilon_0 = 0{\cdot}0119$ and by even less for smaller ϵ_0. The second source of error arises from the non-uniform distribution of the ratios n_2/n_1 in the interval 1/7 to 1. In fact, no ratio exists with a value greater than 0·75. While it is difficult to correct accurately for this effect, the authors do estimate that P_{ϵ_0} is better taken to be $P_{\epsilon_0} = 42{\cdot}8\epsilon_0$ rather than $P_{\epsilon_0} = 39{\cdot}36\epsilon_0$ as was used in Table I.

In light of the preceding discussion we see that the distribution of mean motions among the planets and satellites very definitely deviates from randomness, but it is difficult to say precisely how large this deviation is.

In the present paper we desire a more accurate measure of the preference for near-commensurability among the observed mean motions. We can obtain an improved measure by slightly modifying the anlaysis given by Roy and Ovenden. This modification is intended to include the effect of the non-uniform distribution of the ratios n_2/n_1 in the interval 1/7 to 1. We proceed as follows. For each ratio

n_2/n_1 we form the difference of the two fractions, A_2/A_1 and A'_2/A'_1 which bound it most closely from above and below. We then take as the probability that n_2/n_1 should be within ϵ_0 of one of these integers

$$\mathscr{P}_{\epsilon_0} = \frac{2\epsilon_0}{\dfrac{A_2}{A_1} - \dfrac{A'_2}{A'_1}}.$$

(That is, this is what we would consider this probability to be if n_2/n_1 was a randomly chosen number in this range.) We can repeat this procedure for each of the 46 ratios considered by Roy and Ovenden. Doing this we arrive at the following result. If, aside from their particular distribution between the fractions A_2/A_1, the ratios n_2/n_1 showed no preference for commensurability then the expectation value for the number of near-commensurabilities with $\epsilon \leq \epsilon_0$ would be $E_{\epsilon_0} = 2092\epsilon_0$. The argument just presented has shown the importance of the non-uniform distribution of the ratios n_2/n_1 on the expected number of near-commensurabilities arising from a chance distribution of mean motions. It enables us to supplant Table I by Table II below.

TABLE II

ϵ_0	0·0019	0·0089	0·0059	0·0030	0·0015
E_{ϵ_0}	24·9	18·6	12·3	6·3	3·1
Observed number of pairs of mean motions with $\epsilon \leqslant \epsilon_0$	33	26	20	12	6

In their second paper Roy and Ovenden examined the preference for near-commensurability between the outer satellites of Jupiter and the Sun and between Saturn's satellites, Iapetus and Phoebe. They claimed that these satellites showed a tendency toward near-commensurabilities of the form $n_2/n_1 \simeq 1/A$. That these satellites should exhibit any preference for near-commensurabilities is contrary to the hypothesis that will be expounded in this paper. This hypothesis implies that a preference for near-commensurability should be exhibited only by those satellites for which tidal effects have been sufficient to produce an appreciable change in semi-major axis during the lifetime of the satellite (which is assumed to be $\sim 4 \times 10^9$ yr). Since the tidal effects for the satellites considered here are completely negligible we would not expect them to show any preference for commensurabilities. For the preceding reasons we see that it is important to demonstrate that these satellites do not in fact show any preference for commensurability. This is done using the methods just outlined. We consider a ratio n_2/n_1 for which $1/(A+1) \leq n_2/n_1 \leq 1/A$. Then, if n_2/n_1 is otherwise randomly chosen in this interval, the probability that it would be within ϵ of either $1/(A+1)$ or $1/A$ (where $\epsilon \leq 1/2A(A+1)$) is $2\epsilon A(A+1)$. In Table III we have reproduced the data tabulated by Roy and Ovenden and in addition we have included possible commensurabilities between Phoebe and the Sun and the Moon and the Sun in the last two lines. In the final column we have listed $2\epsilon A(A+1)$. We must note that, as used below, $A' = A$ if $(n_2/n_1) - 1/A' < 0$ and $A' = A + 1$ if $n_2/n_1 - 1/A' > 0$.

Obviously, the probabilities listed in the last column of Table III for the individual pairs of mean motions do not bear out Roy and Ovenden's claim of a preference for near-commensurability. However, the entries for the Sun and the mean of J VIII, J IX, and J XI and for the Sun and the mean of J VI, J VII, and J X do appear to present a strong case for some sort of preference for commensurability.

TABLE III

Satellites	$\frac{n_2}{n_1}$	A'	$\frac{n_2}{n_1}-\frac{1}{A'}$	$2A(A+1)\epsilon$
Sun J VIII	0·17055	6	+0·00388	0·233
Sun J IX	0·17196	6	+0·00529	0·317
Sun J XI	0·15982	6	−0·00684	0·574
Sun and mean of J VIII, J IX, JXI	0·16725	6	+0·00059	0·035
Sun J VI	0·05784	17	−0·0098	0·600
Sun J VII	0·06002	17	+0·00120	0·653
Sun J X	0·05867	17	−0·00015	0·092
Sun and mean of J VI, V VII, JX	0·05883	17	−0·00001	0·006
Sun J XII	0·14564	7	+0·00278	0·234
Iapetus Phoebe	0·14411	7	+0·00125	0·015
Sun Phoebe	0·05125	20	+0·00125	0·950
Sun Moon	0·07480	13	−0·00212	0·772

This case is rapidly dissolved when we investigate the uncertainties in the mean motions used in Table III. These uncertainties, which are mainly produced by the large perturbations of the satellites' orbits by the Sun, cause uncertainties in the individual quantities $(n_2/n_1)-(1/A')$ which are comparable in magnitude to the values given in Table III. These uncertainties may be carried through to give the uncertainties in the entries under the Sun and mean of J VIII, J IX, J XI and under the Sun and mean of J VI, J VII, J X. The apparently significant, small values for $(n_2/n_1)-(1/A')$ in these two cases now appear to be fortuitous. Of course, the arguments just presented, while indicating that the evidence presented in Table III does not constitute a strong case for commensurability, do not rule out the possibility that such commensurabilities may exist.

As an example of the uncertainty in the differences $(n_2/n_1)-(1/A')$, we present the results given by Nicholson (3) for J IX. For the period of J IX he gives 758 ± 25 days. This would imply that the value of $(n_2/n_1)-(1/A')$ for J IX is $0{\cdot}00828\pm0{\cdot}00577$ and that $2\epsilon A(A+1)=0{\cdot}497\pm0{\cdot}346$. These values, while including the ones given by Roy and Ovenden, show clearly how uncertain they are.

It should be noted that if the numbers given by Roy and Ovenden in Table III were free of any uncertainty then it would be very difficult to explain the results listed under the Sun and mean of J VIII, J IX, J XI and under the Sun and mean of J VI, J VII, J X. This is because the members of these two groups of satellites are so small that no significant interactions can exist between them. Hence, even if the mean motions of the individual satellite orbits were oscillating about some orbit which was commensurate with the Sun there would be no explanation for the phases of oscillation in the three satellite orbits to be correlated. Thus, there would be no explanation for the mean of the mean motions of the three satellites to be so much more nearly commensurate with the Sun's mean motion than the mean motion of any of the individual satellites.

In closing this section we might point out that the grouping of Jupiter's satellites' mean motions near values of 6, 7 and 17 times the mean motion of the Sun (as seen from Jupiter) may be of significance when considering the possible capture of these satellites by Jupiter.

2. *The mirror theorem.*—In their second paper, Roy and Ovenden prove that, "If n point-masses are acted upon by their mutual gravitational forces only, and if at a certain epoch each radius vector from the (assumed stationary) centre of mass of the system is perpendicular to every velocity vector, then the orbit of each mass after that epoch is a mirror image of its orbit prior to that epoch". The authors call this theorem the mirror theorem and the special configuration described above is called a mirror configuration. As a corollary to the mirror theorem, they prove a periodicity theorem which states that, "If n point-masses are moving under their mutual gravitational forces only, their orbits are periodic if, at two separate epochs, a mirror configuration occurs".

After proving the preceding theorems, Roy and Ovenden suggest that the frequent occurrence of mirror configurations will cause perturbations on the orbits to undergo frequent reversals so that the disturbances they generate cannot build up to magnitudes so large that they endanger the stability of the motion.

Finally, Roy and Ovenden examined in detail, three of the best cases (i.e., those for which ϵ is smallest) of near-commensurabilities in the solar system. These include three pairs of satellites in Saturn's system: Hyperion and Titan, Enceladus and Dione, and Mimas and Tethys. The values of $(n_2/n_1)-(A_2/A_1)$ for these satellite pairs are $-0{\cdot}000566$, $+0{\cdot}000643$ and $-0{\cdot}000784$ respectively.

Observation provides the following remarkable results: conjunctions of Enceladus and Dione always occur near the perisaturnium of Enceladus. For Titan and Hyperion, the conjunctions always occur near the aposaturnium of Hyperion. For Mimas and Tethys, the relation involves their nodes and the conjunctions of these two satellites oscillate about the midpoint between their two ascending nodes on Saturn's equator plane.

Examination of the motions of these satellite pairs thus reveals that they satisfy the mirror theorem, at least to a first approximation. The nature of this approximation and its dynamical significance will be the subject of the rest of this paper.

3. *Classical perturbation theory.*—In this section a brief outline of the relevant portions of celestial mechanical perturbation theory will be presented. We shall describe the orbit of a satellite of mass m, about a planet of mass M, by the following six elements:

a—is the semi-major axis of the orbit.
e—is the eccentricity of the orbit.
i—is the inclination of the orbit to the planet's equatorial plane.
Ω—is the longitude of the ascending node.
$\tilde{\omega}$—is the longitude of the pericentre.
$\tilde{\omega}=\omega+\Omega$ where ω is the angle between the ascending node and the pericentre.
ϵ—is the mean longitude at epoch.

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To give the position in the orbit, we will use the mean longitude $\lambda = \int_0^t n\,dt + \epsilon$. n, the mean motion of the satellite is equal to $2\pi/P$ where P is the satellite's revolution period. For motion about a spherical planet $n = \sqrt{(GM)}/a^{3/2}$.

For unperturbed motion about a spherical planet a, e, i, Ω, $\tilde{\omega}$ and ϵ are constant. If the motion is perturbed then a, e, i, Ω, $\tilde{\omega}$ and ϵ will, in general, vary with time. For a satellite moving in a total potential $V(F)$ we may define the disturbing function $R(F)$ as $R(F) = V(F) - (GM/r)$. In terms of the disturbing function we may write down the equations of motion for a, e, i, Ω, $\tilde{\omega}$ and ϵ. In what follows, R is considered to be a function of a, e, i, Ω, $\tilde{\omega}$ and λ. In the interest of simplicity we shall neglect powers beyond the second in the satellite's eccentricity and inclination. This allows us to write (4):

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \lambda} \\
\frac{de}{dt} &= \frac{-1}{na^2 e}\frac{\partial R}{\partial \tilde{\omega}} \\
\frac{di}{dt} &= \frac{-1}{na^2 i}\frac{\partial R}{\partial \Omega} \\
\frac{d\epsilon}{dt} &= \frac{-2}{na}\frac{\partial R}{\partial a} \\
\frac{d\tilde{\omega}}{dt} &= \frac{1}{na^2 e}\frac{\partial R}{\partial e} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2 i}\frac{\partial R}{\partial i}
\end{aligned} \tag{1}$$

Perturbations of the first order are obtained by treating a, e, i, Ω, $\tilde{\omega}$ and ϵ as constants on the right hand sides of the perturbation equations, while the mean longitude λ is taken to be a linear function of the time.

The disturbing function due to the action of a satellite with mass m' on one with mass m is given by

$$R = Gm'\left(\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{(r')^3}\right).$$

Coordinates are measured from the centre of the planet. Primes refer to the disturbing satellite. $\Delta^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$.

It can be shown that R is expandable in the following form (5):

$$R = \sum F(a, a', e, e', i, i') \cos T \quad \text{where} \quad T = [h\lambda + h'\lambda' + g\tilde{\omega} + g'\tilde{\omega}' + f\Omega + f'\Omega']$$
$$\text{and } h, h', g, g', f, f' \text{ are integers.} \tag{2}$$

The requirement of rotational invariance gives us the single restriction

$$[h + h' + g + g' + f + f'] = 0. \tag{3}$$

Terms with $h = h' = 0$ give rise to secular changes in $\tilde{\omega}$, Ω and ϵ. The results obtained from first order perturbation theory are only approximate due to treating a, e, i, Ω, $\tilde{\omega}$ and ϵ as constants in the right hand sides of the perturbation equations. If necessary, calculations may be extended to include higher order (the order being measured by the power of m'/M) perturbations. This is done by substituting the results of the first order calculations into the right hand side of the perturbation equations. Higher order perturbation theory is required when treating cases of commensurability involving more than two satellites. One result that we shall quote for later use is Poisson's famous theorem on the invariability of the semi-major axis. This theorem, which is often quoted in discussions of the stability of the solar system, states that there is no secular term, due to gravitational interactions between satellites, in the expression for the semi-major axis, both in the first and second orders of perturbation theory.

Since we shall apply perturbation theory in cases where near-commensurabilities exist, a brief summary of the effects of near-commensurabilities in perturbation theory will be given next. If we write $R=\sum C\cos T$, then the perturbation equations yield, upon integration:

$$\begin{aligned}
\delta_1 a &= \sum + \frac{C_1\cos T}{hn+h'n'} & \delta_1\tilde{\omega} &= \sum + \frac{C_2\sin T}{hn+h\,n'} \\
\delta_1 e &= \sum + \frac{C_1'\cos T}{hn+h'n'} & \delta_1\Omega &= \sum + \frac{C_2'\sin T}{hn+h'n} \\
\delta_1 i &= \sum + \frac{C_1''\cos T}{hn+h'n'} & \delta_1\epsilon &= \sum + \frac{C_2''\sin T}{hn+h'n'}
\end{aligned} \tag{4}$$

(Note: δ_1 denotes first order perturbations.)

From the definition of $\lambda=\int_0^t n\,dt+\epsilon$ we see that

$$\delta_1\lambda = \int_0^t \delta_1 n\,dt + \delta_1\epsilon$$

$$\delta_1\lambda = \sum\left\{\frac{C_1'''\sin T}{(hn+h'n')^2} + \frac{C_2'''\sin T}{(hn+h'n')}\right\}. \tag{5}$$

A near-commensurability implies that one of the terms $h^*n+h'^*n'$ is very small compared to n or n'. From the expressions above, we see that a near-commensurability produces an enhanced amplitude for perturbations of angular frequency $h^*n+h'^*n'$. Since only $\delta_1\lambda$ has the small divisor squared, we see that the principal effect of a near-commensurability will be observed in the perturbations of mean longitude.

4. *Hypothesis of tidal stability.*—We are now in a position to examine the hypothesis of tidal stability as an explanation of the observed commensurabilities. Our considerations will find application to the two-body cases of Enceladus and Dione, Mimas and Tethys, and Hyperion and Titan, as well as to the three-body case of Io, Europa and Ganymede. Other likely candidates for stable commensurabilities, to which these results would also apply, are mentioned in Section 9.

As a start we shall consider a planet surrounded by several satellites which move in orbits of low inclination and eccentricity. We shall make the assumption that the tidal torques on these satellites have produced considerable evolution in their mean motions over a period comparable to the age of the solar system (which we take as four billion years).

Let us make the further assumption that the tidal evolution of the mean motion of each satellite is independent of the other satellites. This independent evolution of mean motions is implied (at least to second order perturbation theory) by

Poisson's theorem on the invariability of the semi-major axis (see Section 2). Since the mean motions of different satellites will, in general, evolve at different rates (if this assumption is correct), the ratios of the mean motions of pairs of satellites will vary with time. In so doing they may occasionally pass through a low order commensurability. However, at any one time, the ratios of the satellites' mean motions will exhibit a tendency for commensurability which is consistent with a random distribution of mean motions. Such a situation would certainly fail to explain the strong tendency for near-commensurate mean motions that is observed among the satellites of the solar system.

If, on the other hand, these near-commensurabilities were stable, then we could account for the large number of observed near-commensurabilities. Suppose, for example, that during the tidal evolution of their mean motions, the ratio of the mean motions of two satellites approaches very closely the ratio of two small integers. If a near-commensurate motion of these two satellites exists, which is stable under further tidal evolution, then the satellites would remain in the near-commensurability rather than merely passing through it. However, the tidal torque on each of these satellites will not be affected by such a near-commensurability of their mean motions. This being the case, in order for the further evolution of the satellite orbitals to proceed without disrupting the near-commensurability of their mean motions, angular momentum must be secularly transferred between the satellites. At first sight this condition might appear to be a violation of Poisson's theorem on the invariability of the semi-major axis. However, this theorem only treats secular terms in the expression for da/dt. That is, only terms which do not depend on the longitudes of the two satellites. In the case of stable near-commensurabilities the terms responsible for the secular transfer of angular momentum are periodic terms, whose periods turn out to be infinite as a consequence of the commensurability. These terms are not dealt with in Poisson's theorem.

We now realize (if we assume that the conjectures of the last paragraph are correct) that the orbits of a pair of near-commensurate satellites will still evolve as the tides feed angular momentum from the planet's spin into the satellites' orbits. However, we shall see that the satellites will share this angular momentum between them in just the correct proportion to keep their mean motions near-commensurate.

The question of which near-commensurabilities are stable will be dealt with next. From the discussion of the previous paragraph we see that a necessary condition for the stability of a near-commensurability is that the direct gravitational forces between the satellites involved are strong enough to be able to distribute the angular momentum fed into the system by the tides in the manner necessary to maintain the commensurability relation. Application of this condition would, in principle, enable us to place bounds on the tidal dissipation in the planets. In practice, however, the small variety of observed near-commensurable mean motions severely limits this possibility.

An examination of the direct gravitational forces between satellites reveals that they decrease rapidly both as the order of the commensurability increases and also as the number of satellites involved increases. This accounts for the low orders of the observed near-commensurabilities (for example, 2 to 1, 4 to 3, 4 to 2, etc.) and for the fact that they only involve two or three satellites. When the direct gravitational forces between satellites are too weak, and angular momentum cannot be transferred between them at a sufficient rate, near-commensurabilities will not be stable and the satellites' mean motions will evolve independently, each at a rate determined by the tidal torque on the single satellite.

As discussed in Section 1, the orbits of several pairs of the best examples of commensurabilities exhibit remarkable regularities. Not only are the mean motions of these satellites near-commensurate, the motions also show a relation between the conjunctions of the satellites and one or more of their orbital elements. Since these relations, relevant to the various satellite pairs, differ only in detail, we shall concentrate our attention on the system of Enceladus and Dione whenever an explicit example is called for.

Denoting the orbital elements of Dione by primes, we can state the observations in the form below (see Section 2 for a definition of these elements).

$$2n' - n - \overline{\frac{d\tilde{\omega}}{dt}} = 0$$

$2\lambda' - \lambda - \tilde{\omega} = V$ where V oscillates about 0° with a small amplitude. Actually, as we shall see later on, V should oscillate about an angle very close to, but not equal to 0°. The second relation, which implies the first one, states that conjunctions of Enceladus and Dione always occur near the perisaturnium of Enceladus. Thus we see that this commensurability relation implies that terms in the disturbing function wth argument $\Phi = 2\lambda' - \lambda - \tilde{\omega}$ are of infinite period. We shall show that when V oscillates about an angle different from 0°, these terms can produce secular changes of the semi-major axis. It may also be noted that what we have here might be considered, not as a near-commensurability of mean motions, but rather as an exact commensurability involving the mean motions of Enceladus and Dione together with the motion of the perisaturnium of Enceladus. The fact that we have a near-commensurability of mean motions is then seen to be merely a consequence of the small size of

$$\overline{\frac{d\tilde{\omega}}{dt}} \Big/ n .$$

Roy and Ovenden attempted to show that these near-commensurate satellite pairs satisfy the hypothesis of their mirror theorem. In the approximation that the inclinations of Enceladus and Dione are neglected and that the eccentricity of Dione is taken as zero, we see that this is the case. Furthermore, when even then eccentricity of Dione is taken into account, it may still be argued that mirror configurations do occur. However, in the other cases of near-commensurability described by Roy and Ovenden, mirror configurations occur only in a first approximation to the actual orbits (for example, when only one eccentricity is taken as non-zero or when both inclinations are considered to be equal, etc.).

5. *Formation of commensurabilities.*—In this section we shall describe the formation of stable commensurabilities. Proofs of some of the statements made in this section will be given in Section 6. Because there is some difference in detail between commensurabilities of the form $n_2/n_1 \simeq A_2/A_1$ where $A_1 = A_2 + 1$ and those for which $A_1 = A_2 + \nu$ where $\nu > 1$ we shall describe them separately. Representatives of both types are to be found in the solar system. For example, the pair of Enceladus and Dione is an illustration of the former ($A_1 = 2$, $A_2 = 1$) while Mimas and Tethys provide an example of the latter* ($A_1 = 4$, $A_2 = 2$).

* For an explanation of the distinction between a 2 to 1 and a 4 to 2 commensurability see Section 8.

Of the other two-body near-commensurabilities (including those for which we may only assume stability) all except the pair of J III and J IV (for which $A_1 = 7$, $A_2 = 3$) are of the form $A_1 = A_2 + 1$.

An idealized model which contains all of the relevant features will be considered next. Although the situation envisaged in this idealized model does not correspond to any actual near-commensurability it allows us to present the essential features of near-commensurabilities without any unnecessary additions. We imagine two satellites with masses m and m' orbiting about a planet of mass M. The masses of these satellites obey the relations $m/M \ll 1$, $m'/M \ll 1$ and $m'/m \ll 1$ while their orbits satisfy the restrictions $i = i' = 0$, $e = 0$, $e' \ll 1$, $a < a'$. The restriction $m'/m \ll 1$ allows us to neglect perturbations of m by m'. We shall also assume that the initial periods of revolution are at least as long as the planet's rotation period so that the tides feed angular momentum into the satellites' orbits. Since $m'/m \ll 1$ and $a < a'$ we shall neglect the tidal evolution of m''s orbit. We define $\overline{d\tilde{\omega}'/dt}$ as the observed secular rate of change of $\tilde{\omega}'$ while $(d\tilde{\omega}'/dt)_s$ is that part of $\overline{d\tilde{\omega}'/dt}$ which is produced by secular terms in m''s disturbing function. This distinction between $\overline{d\tilde{\omega}'/dt}$ and $(d\tilde{\omega}'/dt)_s$ is a non-trivial one if the satellites are part of a near-commensurability. For in that case some argument (for example, $2\lambda' - \lambda - \tilde{\omega}'$) is constant. Then periodic terms in the disturbing function having this argument will also produce secular motions of $\tilde{\omega}'$. These are included in $\overline{d\tilde{\omega}'/dt}$ but not in $(d\tilde{\omega}'/dt)_s$.

The first case we shall discuss is one in which initially $2n' - n - (d\tilde{\omega}'/dt)_s < 0$. We shall neglect the possibility of near-commensurabilities other than the one for which $2n' - n - \overline{d\tilde{\omega}'/dt} = 0$. As m's orbit expands due to the addition of angular momentum by the tides, $2n' - n - (d\tilde{\omega}'/dt)_s$ will approach zero through negative values. As this occurs m will force an ever increasing eccentricity onto the orbit of m'. The pericentre which corresponds to this forced eccentricity will satisfy the relation $2n' - n - (\overline{d\tilde{\omega}'/dt}) = 0$. When this eccentricity becomes of sufficient size m's orbit will no longer expand independently of m''s. Indeed, as we shall discover, m will begin to feed angular momentum into m''s orbit at such a rate that the approach of $2n' - n - (d\tilde{\omega}'/dt)_s$ to zero is slowed down. In fact if the tidal torque on m is not too large this state of affairs will persist for many times the age of the solar system. $2n' - n - (d\tilde{\omega}'/dt)_s$ will continue to approach zero but always decelerating, and the forced eccentricity of m''s orbit will continue to increase. In this manner a stable near-commensurability is formed.

We may now modify this argument so that it applies to a case where initially $3n' - n - 2(d\tilde{\omega}'/dt)_s$ is slightly less than zero. This time as m moves out it does not force an appreciable eccentricity on the orbit of m'. Hence, angular momentum cannot be secularly transferred from m to m' and m will move out without affecting m'. Eventually when $3n' - n - 2(d\tilde{\omega}'/dt)_s$ has become very close to zero a stable near-commensurability may be formed (if the tidal torque on m is not too great). In this case $\overline{d\tilde{\omega}'/dt}$ will be very nearly equal to $(d\tilde{\omega}'/dt)_s$ and the eccentricity of m''s orbit will be its free eccentricity. Furthermore, once this commensurability has been established $3n' - n - 2(d\tilde{\omega}'/dt)_s$ will remain essentially constant and will not continue to change as $2n' - n - (d\tilde{\omega}'/dt)_s$ did in the case of the two to one near-commensurability.

Finally, we must mention that similar commensurabilities involving the satellites' inclinations and ascending nodes can be formed. However, no analogue to the $A_1 = A_2 + 1$ commensurabilities can exist which involves inclinations and nodes. Only cases for which $A_1 \geq A_2 + 2$ are possible when these elements are involved.

6. *Analysis of near-commensurabilities.*—In this section proofs for some of the statements made in the preceding section are provided. Throughout, we shall neglect the effects of m' on m and the action of the tides on m' as well. The first case to be treated will be the two to one near-commensurability.

In our stability analysis we shall neglect all periodic terms in m''s disturbing function except those having $2\lambda' - \lambda - \tilde{\omega}'$ as their argument. It is easy to see why terms of this argument might produce instability. By hypothesis, this argument has zero secular rate of change. Substituting terms with this argument into the perturbation equations we see that they produce secular changes in the orbital elements. It might be suspected that these secular changes will disrupt the near-commensurability. However, we shall prove that under certain conditions this does not happen. Terms in the disturbing function whose arguments are integer multiples of Φ will be neglected since their coefficients are small than those of the terms with argument Φ by e' raised to the power of the absolute value of the corresponding integer multiple minus one. All periodic terms whose arguments are not multiples of Φ produce only small magnitude, short period perturbations of m''s orbit, since their amplitudes are not enhanced by integration (see Section 3). Finally, Poisson's theorem on the invariability of the semi-major axis tells us that we need not worry about secular terms in the expression for the rate of change of the semi-major axis.

The term in m''s disturbing function with argument Φ is given to first order in e' by

$$R'\Phi = \frac{e'\,Gm}{2a'}\,(C(\alpha))\cos\Phi \tag{6}$$

where $\alpha = a/a'$ and $C(\alpha)$ is a sum of Laplace coefficients which may be expressed in terms of elliptic integrals. We shall need the value of $C(\alpha)$ when we apply our results to specific satellite systems. At present we shall merely use the result that $C(\alpha)$ is positive.

Using equations (1) and (6) we arrive at the following expressions for the rates of change of the orbital elements when tidal forces are absent.

$$\begin{aligned}
\frac{de'}{dt} &= -\frac{1}{n'a'^2e'}\frac{\partial R'}{\partial\tilde{\omega}'} = -\left(\frac{m}{M}\right)\frac{n'C(\alpha)}{2}\sin\Phi \\
\frac{d\tilde{\omega}'}{dt} &= \frac{1}{n'a'^2e'}\frac{\partial R'}{\partial e'} = \left(\frac{m}{M}\right)\frac{n'C(\alpha)}{2e'}\cos\Phi + \left(\frac{d\tilde{\omega}'}{dt}\right)_s \\
\frac{dn'}{dt} &= -\frac{3}{2}\frac{n'}{a'}\frac{da'}{dt} = -\frac{3}{a'^2}\frac{\partial R'}{\partial\lambda'} = 3e'\left(\frac{m}{M}\right)n'^2\,C(\alpha)\sin\Phi \\
\frac{d\epsilon'}{dt} &= -\frac{2}{n'a'}\frac{\partial R'}{\partial a'} = +e'\left(\frac{m}{M}\right)n'\left[C(\alpha) + \alpha\frac{dC(\alpha)}{d\alpha}\right]\cos\Phi.
\end{aligned} \tag{7}$$

We define $V = 2\lambda' - \lambda = \Phi + \tilde{\omega}'$.

$$\frac{d^2V}{dt^2}=\frac{2dn'}{dt}-\frac{2d^2\epsilon'}{dt^2}=6e'\left(\frac{m}{M}\right)n'^2\,C(\alpha)\sin\Phi$$
$$-2e'\left(\frac{m}{M}\right)n'\left[C(\alpha)+\alpha\frac{dC(\alpha)}{d\alpha}\right]\sin\Phi\,\frac{d\Phi}{dt}.$$

It is a trivial matter to verify that a zero order solution of the above equations is given by

$$\Phi=\Phi_0=180^\circ,\quad e'=e_0'=\left(\frac{m}{M}\right)\frac{C(\alpha)}{2}\frac{n'}{(n-2n'+(d\tilde{\omega}'/dt)_s)}\;\overline{d\tilde{\omega}_0'/dt}=2n'-n. \tag{8}$$

This solution corresponds to a situation in which the free eccentricity is equal to zero and the total eccentricity has just its forced value e_0'.

Next we consider first order deviations from this solution. Setting

$$\Phi=\Phi_0+\Phi_1,\quad e'=e_0'+e_1'\quad\text{and}\quad\tilde{\omega}'=\tilde{\omega}_0'+\tilde{\omega}_1'$$

the relevant equations take the following form:

$$\begin{aligned}
\frac{de_1'}{dt}&=\left(\frac{m}{M}\right)\frac{n'C(\alpha)}{2}\Phi_1\\
\frac{d\tilde{\omega}_1'}{dt}&=\left(\frac{m}{M}\right)\frac{n'C(\alpha)}{2e_0'^2}e_1\\
\frac{d^2V_1}{dt^2}&=-6e_0'\left(\frac{m}{M}\right)n'^2\,C(\alpha)\Phi_1+2e_0'\left(\frac{m}{M}\right)n'\left[C(\alpha)+\alpha\frac{dC(\alpha)}{d\alpha}\right]\Phi_1\frac{d\Phi_1}{dt}\\
\frac{d^2\Phi_1}{dt^2}&=\frac{d^2V_1}{dt^2}-\frac{d^2\tilde{\omega}_1'}{dt^2}.
\end{aligned}\tag{9}$$

In most cases we may safely neglect the $\Phi_1\,d\Phi_1/dt$ term in the equation for d^2V_1/dt^2. We shall drop it for the time being and later formulate the condition under which it is negligible. In writing down equations (9) we have assumed $\Phi_1\ll 1$ although the analysis is not very different in cases with Φ_1 of order unity.

From the equations for de_1'/dt and $d\tilde{\omega}_1'/dt$ we obtain

$$\begin{aligned}
\frac{d^2\tilde{\omega}_1'}{dt^2}&=\left[\left(\frac{m}{M}\right)\frac{n'C(\alpha)}{2e_0'}\right]^2\Phi_1\\
\frac{d^2\Phi_1}{dt^2}&=-\left\{6e_0'\left(\frac{m}{M}\right)n'^2\,C(\alpha)+\left[\left(\frac{m}{M}\right)\frac{n'\,C(\alpha)}{2e_0'}\right]^2\right\}\Phi_1=-\gamma^2\Phi_1.
\end{aligned}\tag{10}$$

Thus Φ_1 is seen to satisfy a simple pendulum equation (in the small amplitude approximation). Solving this equation we find

$$\Phi_1=\theta\sin\gamma t. \tag{11}$$

The neglect of the $\Phi_1\,d\Phi_1/dt$ term in equations (9) is now seen to be equivalent to neglecting $\theta\gamma/n$. This quantity may be computed in cases of observed commensurabilities.

The next question to be dealt with is the stability of this near-commensurability when tidal forces are present. In this section we shall denote the rate of change of m's mean motion due to the tidal forces by dn_T/dt. In Section 11 this quantity is related to the properties of the satellite and its planet. Rewriting equation (10) to include tidal effects we get

$$\frac{d^2\Phi_1}{dt^2}=-\gamma^2\sin\Phi_1-\frac{dn_T}{dt}. \tag{12}$$

If the tidal term is sufficiently small then

$$\Phi_1 = \theta \sin \gamma t - \frac{1}{\gamma^2} \frac{dn_T}{dt} \tag{13}$$

(we have again assumed $\Phi_1 \ll 1$).

Hence the only effect on Φ_1 produced by the tidal torque, is a phase shift. Thus "weak" tidal forces do not disrupt the relation $2n' - n - (d\tilde{\omega}'/dt) = 0$. To consider the stability of the near-commensurability in mean motions we must evaluate $(d^2V_1/dt^2) = (2dn'/dt) - (dn/dt)$. In the absence of mutual interactions between the satellites this quantity would be equal to $-dn_T/dt$ and the near-commensurability in mean-motions would not persist. However, using equations (9) and (13) we find

$$2\frac{dn'}{dt} - \frac{dn}{dt} = -6e_0'\left(\frac{m}{M}\right) n'^2 C(\alpha)\, \theta \sin \gamma t - \frac{\dfrac{dn_T}{dt}}{\left\{1 + \left[24e_0'3 \Big/ \left(\dfrac{m}{M}\right) C(\alpha)\right]\right\}}. \tag{14}$$

From this equation we see that apart from an oscillating term the forced eccentricity of m''s orbit allows a transfer of angular momentum from m to m' which decreases

$$\left|2\frac{dn'}{dt} - \frac{dn}{dt}\right|$$

from

$$\left|\frac{dn_T}{dt}\right|$$

to

$$\frac{\left|\dfrac{dn_T}{dt}\right|}{\left\{1 + \left[24e_0'^3 \Big/ \left(\dfrac{m}{M}\right) C(\alpha)\right]\right\}}.$$

Since we know that e_0' increases as $2n' - n - (d\tilde{\omega}'/dt)_s$ goes to zero, this tells us that

$$\left|2\frac{dn'}{dt} - \frac{dn}{dt}\right|$$

continues to decrease as $2n' - n - (d\tilde{\omega}'/dt)_s$ approaches zero. This analysis is only valid for $e_0' \ll 1$ so that the details of this process cannot be followed indefinitely. However, it seems likely that if the eccentricity of m''s orbit becomes large enough it may cross or nearly cross the orbit of m. In this case a near collision or a physical collision between these satellites would become likely. This might lead to the capture of m' by m. In some cases, however, e_0' will remain small for many times the age of the solar system and the analysis just presented will be quite adequate to describe the commensurability for all times less than this.

The analysis just presented for the case $A_1 = 2$, $A_2 = 1$ can easily be modified to fit the case $A_1 = 3$, $A_2 = 1$. The full details of the proof in this case will not be written out. However, differences between this case and the preceding one will be pointed out. If we set $\Phi = 3\lambda' - \lambda - 2\tilde{\omega}'$

$$R'\Phi = \frac{e'^2 Gm}{2a'} [B(\alpha)] \cos \Phi. \tag{15}$$

It is the extra power of e' in R'_Φ that makes this case different from the former one. In particular, an examination of equations (1) shows us that no large forced eccentricity is produced in this case. However, given a free eccentricity e_0' a stable near-commensurability can be formed. The solution which corresponds to this case is easily seen to take the following form:

$$\Phi_0 = 180°, \quad \frac{2\overline{d\tilde{\omega}_0'}}{dt} = 3n' - n = 2\left(\frac{d\tilde{\omega}'}{dt}\right)_s + \left(\frac{m}{M}\right) n' B(\alpha)$$

$$\Phi_1 = \theta \sin \gamma t - \frac{dn_T}{dt} \Big/ \gamma^2 \tag{16}$$

$$\frac{d^2 V_1}{dt^2} = 3\frac{dn'}{dt} - \frac{dn}{dt} = \frac{d^2\Phi_1}{dt^2} = -\gamma^2 \theta \sin \gamma t.$$

Thus we find that Φ_1 oscillates about $-(dn_T/dt)/\gamma^2$. $3(dn'/dt) - (dn/dt)$ also oscillates but in this case it has no constant part as it did in the preceding case. de_1'/dt has a constant part equal to $-(mn'B(\alpha)/M2)(dn_T/dt)/\gamma^2$. This tells us that the free eccentricity is increasing due to the mutual interaction of the tides and m.

7. *Explanation of the results of Sections* 5 *and* 6.—The discussion in this section is intended to provide physical explanations of the results obtained in the preceding two sections.

We shall begin by considering the two to one near-commensurability which was described in these sections. For this near-commensurability we have seen that conjunctions between the two satellites take place when m' is close to apocentre. In the first instance we shall neglect tidal torques and investigate the stability of $\Phi = 2\lambda' - \lambda - \tilde{\omega}'$ about 180°. Once again we use the restrictions placed on m, m' and their orbits in Section 4. We suppose, for the sake of argument, that Φ is initially chosen slightly greater than 180° and $d\Phi/dt$ is taken to be zero. Then the next conjunction will take place after m' has passed through apocentre. This produces a net transfer (over an entire revolution of m') of angular momentum from m to m'. To see this we must observe that between an opposition and the following conjunction m is taking angular momentum from m', while after conjunction until the next opposition m is adding angular momentum to m'. By symmetry, if the conjunction occurs when m' is at apocentre, the net transfer of angular momentum would be equal to zero. However, if m' is slightly past apocentre when this conjunction takes place, then the asymmetry in the relative separations of the satellites, at equal times before and after conjunction, will imply that a net transfer of angular momentum will take place. Rather than take the space necessary to prove it, we shall merely state that $\Phi - 180°$ positive implies a net gain of angular momentum by m', while $\Phi - 180°$ negative implies a net loss of angular momentum by m'. This net transfer of angular momentum to m' (we have chosen $\Phi - 180°$ positive initially), increases m''s period and at the next conjunction m' will be closer to apocentre than at the previous one. Thus Φ will have been decreased. Similarly if Φ becomes smaller than 180° there will be a net loss of angular momentum by m'. In this case Φ will be increased toward 180°. The process we have just described is the one which stabilizes Φ about 180° and produces the librations whose angular frequency is given by γ in equation (11). A similar argument will show us that 0° is not a stable value for Φ in this case.

When tidal forces are present, m's period is steadily lengthening. This tends to produce conjunctions at values of Φ slightly in excess of 180°. Hence, as we have just seen, m will, on the average, (over many libration periods) feed angular momentum into m''s orbit. This will enable m to push m' out ahead of itself at just the correct rate needed to maintain their near-commensurability.

The stability of the three to one near-commensurability has a similar origin. However, in this case conjunctions occur alternately near m''s apocentre and its pericentre. The net transfer of angular momentum (when $\Phi \neq 0$) which arises due to the asymmetry before and after each conjunction is of opposite sign for conjunctions occurring near m''s apocentre and pericentre. To first order in e', the two contributions cancel each other. However, there is still a net effect to order e'^2 which insures stability in this case. It is just this cancellation, which occurs in all cases except $A+1$ to A commensurabilities, that prevents the production of large forced eccentricities in this case.

We turn next to the discussion of the other major difference between the three to one and the two to one near-commensurability. In the case of the two to one near-commensurability it is the forced eccentricity of m''s orbit which is responsible for the stability of Φ. When this forced eccentricity is small, the pericentre associated with it behaves as though it had a very small "inertia". That is, this pericentre is able to adjust itself so that conjunctions of the satellites oscillate about a point which is very close to apocentre. When this is the case, the tidal stability described above will not be present and m will move out without pushing m' ahead of it. However, as the forced eccentricity of m''s orbit increases its pericentre's effective "inertia" increases as well. The forced eccentricity now behaves more like its free counterpart (in the case of the three to one commensurability) and m will begin to feed angular momentum to m'. The behaviour just described explains equation (14) which shows how the difference $(2dn'/dt)-(dn/dt)$ decreases as the forced eccentricity e_0' increases. This behaviour is to be contrasted with that expressed by equations (16) for the case of the three to one commensurability. Here $(3dn'/dt)-(dn/dt)$ has no constant part once the commensurability has been established.

8. *Observed stable near-commensurabilities.*—In this section we shall indicate how the results derived in Section 6 apply to those observed near-commensurabilities for which tidal stability may be explicitly demonstrated.

(*a*) Enceladus and Dione

The pertinent data are listed below.

TABLE IV

Enceladus	Dione
$P=2\pi/n=1\cdot37028$ days	$P'=2\pi/n'=2\cdot73691$ days
$e=0\cdot0045$	$e'=0\cdot0021$
$i=0\cdot0°$	$i'=0\cdot0°$
$\overline{d\varpi/dt}=123\cdot43°$ per year	$\overline{d\varpi'/dt}=30\cdot74°$ per year
$M/m=8\times10^6$	$M/m'=5\cdot5\times10^5$

This commensurability involves the argument $\Phi=2\lambda'-\lambda-\tilde{\omega}$. It is an example of a two to one near-commensurability. Φ oscillates about 0° with a period of approximately 12 years and an amplitude of about 20′. The phase shift in Φ due to the tides has a value of about $4 \times 10^{-3}/Q$ radians and is too small to be observed since Q for Saturn is of the order of 10^4 to 10^5 (see Section 12).

(*b*) Titan and Hyperion

The pertinent data are listed below

TABLE V

Titan	Hyperion
$P = 2\pi/n = 15\cdot945452$ days	$P' = 2\pi/n' = 21\cdot27666$ days
$e = 0\cdot0290$	$e' = 0\cdot104$
$i = 0\cdot3°$	$i' = 0\cdot5°$
$\overline{d\varpi}/dt = 0\cdot0512°$ per year	$\overline{d\varpi'}/dt = 18\cdot652°$ per year
$M/m = 4\cdot15 + 10_0$	$M/m' = 5 + 10_2$

This system is similar to that of Enceladus and Dione. However, the analysis is complicated by the large mass of Titan which forces a correspondingly large eccentricity onto the orbit of Dione. This makes it necessary to keep many terms in the disturbing function of Hyperion in order to get accurate numerical results. However, the essential details of this system are quite straightforward.

$$4n' - 3n - \frac{\overline{d\tilde{\omega}'}}{dt} = 0$$

and the angle $\Phi = 4\lambda' - 3\lambda - \tilde{\omega}'$ oscillates about 180°. Unfortunately, in this case the phase shift in Φ due to the tidal forces is even smaller than that for Enceladus and Dione and offers no hope for direct observations.

(*c*) Mimas and Tethys

The pertinent data are listed below.

TABLE VI

Mimas	Tethys
$P = 2\pi/n = 0\cdot942422$ days	$P' = 2\pi/n' = 1\cdot887802$ days
$e = 0\cdot0201$	$e' = 0\cdot0$
$i = 1\cdot5°$	$i' = 1\cdot1°$
$\overline{d\Omega}/dt = 365\cdot2°$ per year	$\overline{d\Omega'}/dt = 72\cdot2°$ per year
$M/m = 1\cdot5 + 10^7$	$M/m' = 8\cdot7 + 10^5$

This commensurability involves the orbits' nodes instead of their pericentres. The conjunctions of these two satellites oscillate about the midpoint of their two ascending nodes on Saturn's equator plane. The commensurability between Mimas and Tethys is properly classified as a four to two rather than a two to one near-commensurability. This distinction arises because the term in the disturbing function which has constant argument is proportional to

$$\cos(4\lambda' - 2\lambda - \Omega' - \Omega).$$

In this case the inclinations of these satellites' orbits are free inclinations rather than forced ones. Furthermore, as we would expect from the discussion of Section 6 $\overline{(d/dt)(\Omega' + \Omega)}$ is very close to $[(d/dt)(\Omega + \Omega')]_s$ and the discrepancy between these numbers is within the margins of observational error. Once again the phase shift in Φ produced by the tides is unobservably small.

9. *Two-body near-commensurabilities among Jupiter's Galilean satellites.*—In addition to taking part in a three-body commensurability (which is discussed

in the following section) Jupiter's Galilean satellites are involved in several two-body near-commensurabilities. Denoting the mean motions of Io, Europa, Ganymede and Callisto by n_1, n_2, n_3 and n_4, we can write the relations satisfied by their mean motions in the following form:

$$\frac{n_2}{n_1} - \frac{1}{2} = -0{\cdot}001817$$

$$\frac{n_3}{n_2} - \frac{1}{2} = -0{\cdot}003647$$

$$\frac{n_3}{n_1} - \frac{1}{4} = -0{\cdot}002725$$

$$\frac{n_4}{n_3} - \frac{3}{7} = +0{\cdot}000128$$

Due to the large masses of these satellites and to the presence of commensurabilities, the mutual perturbations of the orbits of these satellites are both large and complicated. For this reason, any attempt to prove that these two-body near-commensurabilities are tidally stable would require a considerably more involved analysis than that presented in Section 6. Fortunately, while a more detailed analysis is lacking, there do exist two related pieces of evidence which strengthen our belief in the stability of these near-commensurabilities. The first piece of evidence is the remark of Griffin (**6**) that for the first three Galilean satellites, the inner of the pair is near perijove and the outer near apojove whenever a conjunction takes place. Since relations of this type are just what we have found to occur in tidally stable commensurabilities this remark strongly suggests that these commensurabilities are stable. The second piece of evidence favouring stability is the following: If we consider only the pair of Io and Europa and neglect the presence of Ganymede and Callisto then the arguments presented in Section 6 would tell us that the outer satellite (Europa) should be at apojove and the inner satellite should be at perijove whenever conjunctions occurred. Furthermore, $2n' - n$ would be less than zero. The three restrictions just enumerated are necessary conditions for a stable commensurability to exist between Io and Europa. Since all three are satisfied in this case and in the case of Europa and Ganymede as well, we may conclude that while no proof of the stability of these two-body near-commensurabilities has been presented there is good reason to believe that they are stable.

10. *The Laplace relation.*—The three-body commensurability between Io, Europa and Ganymede was first discussed by Laplace and since then has been investigated by many authors (**7**). In this system the commensurability relation involves the mean-motions of these three bodies. (They will be denoted by n_1, n_2 and n_3.) $\lambda_1 - 3\lambda_2 + 2\lambda_3$ oscillates about 180° with a very small amplitude. The proof of the stability of this relation, under the action of tidal forces, involves second order perturbation theory; otherwise it goes through in exactly the same manner as the stability proofs in the two-body cases. The stability for this case was known to Laplace (**8**). In this case also, the phase shift in Φ is very small. It can be estimated to be approximately $6 \times 10^{-2}/Q$ radians.

In the past, several authors have placed lower limits on the value of Q (for a definition of Q see Section 11) of Jupiter. This was accomplished by noting that observations of Io (Jupiter I) gave no secular acceleration, to within the observational accuracy. Lower bounds on Q as high as 10^4 have been set by this method. It is interesting to see how the tidal stability of the commensurability relation changes our estimates of this lower bound. The amount of angular momentum which the tides transfer from the spin of Jupiter into orbital angular momentum of the satellites is not affected by the commensurability. However, in order to maintain the relation $n_1 - 3n_2 + 2n_3 = 0$, this angular momentum must be shared among Jupiter's satellites in a special way. Three possibilities will be considered. They differ in their assumptions about the stability of the near-commensurabilities between pairs of the Galilean satellites. These near-commensurabilities may be written as $2n_2 - n_1 = 0$, $2n_3 - n_2 = 0$ and $7n_4 - 3n_3 = 0$. These may or may not be stable relations (see Section 9). If we assume the stability of the first two, two-body commensurabilities, and the three-body one, then the lower bound of Q must be decreased by a factor of about 5. If we assume all the above commensurabilities are stable, then we must decrease this bound by a factor of approximately 7·4. If only the three-body commensurability is assumed stable, we again get a reduction of about a factor 5.

11. *The success of the hypothesis of tidal origin for the commensurabilities.*—We have proposed that commensurabilities are a consequence of the tidal evolution of satellite orbits. If this hypothesis is to be tenable, the tidal evolution of the satellite systems involved must have been appreciable. If we assume that these satellites have existed for a time comparable to the age of the solar system (which we take as 4×10^9 years) we can examine this question of tidal evolution in some detail. The rate of change of a satellite's mean motion due to tidal friction is given by (**9**)

$$-\frac{dn_T}{dt} = \frac{27n^2}{4}\left(\frac{m}{M}\right)\left(\frac{R}{a}\right)^5 \frac{1}{\left(1+\frac{19\mu}{2g\rho R}\right)Q}. \tag{17}$$

Here we have used a homogeneous sphere model of our planet. ρ, g, R, μ and Q are its density, surface gravity, radius, rigidity and specific dissipation function respectively, Q is defined as 2π times the peak energy stored in a cycle divided by the energy dissipated over the cycle (for a more complete discussion of Q and its relation to the phase lag of the tides see reference (**10**)). Jeffreys (**11**) has tabulated $d\xi/dt$ (ξ is defined by $n = n_0\xi^{-3}$ where n_0 is the present value of the mean motion) for all satellites in the solar system. From his list we have selected those values of $d\eta/dt = d\xi/dt(1 + 19\mu/2g\rho R)Q$ which are greater than $10^{-17}\,\text{sec}^{-1}$. These quantities have been listed in Table VI.

From the results of previous sections we now know that stable commensurabilities exist between Mimas and Tethys, Enceladus and Dione, Titan and Hyperion and Io, Europa and Ganymede. Furthermore, it is strongly suspected that Io and Europa, Europa and Ganymede, and Ganymede and Callisto form stable two-body commensurabilities while Dione, Rhea and Titan may possibly take part in a stable three-body commensurability.

From Table VI (and remembering that Ganymede is being pushed out by Io and Europa) we see that in all the above-mentioned commensurabilities, excepting the system of Titan and Hyperion, at least one of the satellites involved has $d\eta/dt \geq 10^{-14}\,(\mathrm{sec})^{-1}$. Turning this argument around, we may ask the following question: what percentage of all satellites, for which $d\eta/dt \geq 10^{-14}\,\mathrm{sec}^{-1}$ is involved in a commensurability relation? Inspection of Table VI tells us that every satellite, except Phobos, which fulfils this criterion is part of at least one definitely stable commensurability. Furthermore, it is no surprise that Phobos is not involved in a stable commensurability with Deimos, the other Martian satellite. This is because the mutual gravitational interactions between them are very weak as a consequence of their small size (about 8 and 4 km in radius, respectively).

TABLE VII

Satellite	$d\eta/dt = d\zeta/dt(1 + 19\mu/2g\rho R)Q(\mathrm{sec})^{-1}$
Phobos	2.6×10^{-14}
Jupiter V	7.0×10^{-15}
Io	4.9×10^{-13}
Europa	1.6×10^{-14}
Ganymede	2.5×10^{-15}
Callisto	4.0×10^{-17}
Mimas	4.0×10^{-14}
Enceladus	1.6×10^{-14}
Tethys	3.2×10^{-14}
Dione	1.0×10^{-14}
Rhea	2.0×10^{-15}
Titan	6.9×10^{-16}
Ariel	2.3×10^{-15}

Looking at the Jovian commensurabilities, we see that the only condition that must be satisfied, in order that a tidal origin theory be tenable, is that the tidal evolution of Io's orbit has been appreciable. This being the case Io would have pushed out the orbits of the next two Galilean satellites as a consequence of their commensurability.

In the Saturnian system, we have three stable two-body near-commensurabilities. In this case the condition that a tidal theory of origin be tenable is a far stricter one than for the Jovian system. Due to the strong dependence of a satellite's tidal acceleration on the distance from its primary the present value of $d\xi/dt$, for a satellite whose orbit has undergone appreciable tidal evolution, is very closely specified. We can express this behaviour quantitatively by integrating equation (17) to obtain:

$$\frac{n}{n_0} = \left[1 \Big/ \left(1 - 13\frac{d\xi}{dt}T\right)\right]^{3/13}. \tag{18}$$

In this equation T is the time elapsed since the satellite's mean motion was n. If we consider two satellites whose values of $d\xi/dt$ differ by a factor of 10, then assuming the maximum possible change in n for the satellite with greater $d\xi/dt$ we see that $n/n_0 = (10/9)^{3/13} = 1.023$ for the other satellite. Thus the mean motion of the satellite with smaller $d\xi/dt$ was only changed by 2 per cent. This example illustrates the severe restriction our tidal hypothesis places on the present values of $d\xi/dt$. For if all the satellites involved in commensurabilities are assumed to be of the same age, and in addition the orbit of at least one satellite in each commensurate pair is assumed to have undergone appreciable tidal evolution, then the present values of $d\xi/dt$ for these satellites should be very similar. In the Jovian system this requirement presents no test of the theory since it is only necessary that Io's orbit should have been appreciably expanded by the direct action of the tides. In Saturn's system however, the situation is quite different. Here we have three distinct cases of two-body near-commensurabilities. For the pairs of Enceladus and Dione and Mimas and Tethys we find $d\xi/dt$ for the four satellites varying by a factor four. Given the uncertainties involved in taking Q independent of the frequency and amplitude of the tides, as well as the assumption that these satellites are all of the same age, this agreement is remarkably close and offers some evidence for our hypothesis. The third near-commensurate pair in Saturn's system, that of Titan and Hyperion, presents quite another story. By far the larger values of $d\eta/dt$ ($6\cdot9 \times 10^{-16}\,\mathrm{sec}^{-1}$) belong to Titan. However, this value is very small when compared to the values of Enceladus, Dione, Mimas and Tethys. In this case we are forced to admit that this near-commensurability is due to chance or, in other words, to a particular set of initial conditions. It could not have arisen from any conceivable tidal evolution. Of course, once it had been formed it would not have been broken down by the tides since Titan has Hyperion very firmly locked into this commensurability. At first sight this exception to our hypothesis might seem to spell its downfall. Fortunately, this does not turn out to be the case. Using the results quoted in Section 1, we may observe that in a random sample of 46 ratios of mean motions, the expected number, having ϵ smaller than that for the system of Titan and Hyperion is about one. Thus we may safely conclude that this commensurability arose by chance.

Thus far in this section we have shown that the stable commensurabilities involve those satellites for which $d\xi/dt$ is greatest (with the exception of the Titan–Hyperion commensurability). This behaviour is in complete accord with what would be expected if these commensurabilities were of tidal origin. Further more, if we subtract from the listed number of observed near-commensurabilities, those which we believe to have been formed by the tides, the Table II will show no significant preference for near-commensurability among the remaining ratios of mean motions. This agreement is even more striking when we take into account the possibility of "multiple counts" discussed in Section 1.

We may conclude this section by stating that insofar as they have any leaning on our hypothesis, the observations fully tend to confirm it.

12. *Estimates of Q.*—Information obtained about the Q's of Jupiter and Saturn is summarized in this section. Strictly speaking, all numbers refer to the combination $Q(1+19\mu/2g\rho R)$. However, $(1+19\mu/2g\rho R)$ is likely to be very close to unity for these planets. For this reason we shall always refer to Q in what follows.

A method of bounding Q from below that immediately suggests itself is the following. We assume that all the inner satellites of these planets were formed (with essentially their present masses) at about the same time that the Earth was (about 4×10^9 years ago). Then assuming, in addition, the Q was constant during this period we may apply equation (18) which tells us each satellite's value of n (as a function of Q) 4×10^9 years ago. We then ask for the smallest value of Q, consistent with none of the satellites being at its planet's surface less than 4×10^9 years ago. This gives us the first lower bound for Q. Any bound derived in this manner is open to some doubt owing to the presence of cooperative interactions between the satellites in question. However, bearing this in mind we have calcuated this bound using the values appropriate to the satellite of each planet with the greatest value of $d\xi/dt$ (see Table VI). In this manner we arrive at the results listed below.

$$Q \text{ for Jupiter} \geqslant 7 \cdot 6 \times 10^5 \text{ (based on } d\xi/dt \text{ for Io)}$$

$$Q \text{ for Saturn} \geqslant 6 \cdot 4 \times 10^4 \text{ (based on } d\xi/dt \text{ for Mimas).}$$

However, if we use the additional information (see Section 10) that Io is involved in at least one stable commensurability, then this bound for Q is lowered by a factor of between 5 and 7·5.

It is perhaps worth mentioning that if the procedure described above is applied to the Earth–Moon system a puzzling result is obtained. In this case $d\xi/dt$ is known from observation. However, using the observed value of $d\xi/dt$ is equation (18) we find that the Moon should have been near the Earth approximately $1 \cdot 6 \times 10^9$ years ago. Possible resolutions of this apparent paradox are discussed by MacDonald (**12**). They involve the rejection of one or both of the assumptions made above. Firstly, that the value for the Q of the Earth has been sensibly constant during the Earth's history and secondly, that the Moon, with essentially its present mass, has been in an Earth orbit for approximately the lifetime of the Earth.

The second method of deriving a lower bound for the Q of Jupiter is by direct observation of the secular acceleration of Io. The absence of any observable secular acceleration has produced a lower bound of 10^4 in past discussions. As mentioned in Section 10 these observations must be interpreted in light of the commensurabilities among Jupiters' Galilean satellites. If this is done we arrive at a lower bound between $(1 \cdot 5 - 2) \times 10^3$ for the Q of Jupiter. This bound, unlike the previous one, does not depend on any assumptions concerning the ancient history of these satellites.

Upper bounds for the Q's of Jupiter and Saturn may be estimated if we make the additional assumption of the tidal origin of commensurabilities. This assumption implies that those satellites which are involved in commensurabilities have had considerable evolution of their mean motions. This conclusion tells us that the first set of lower bounds for Q, derived in this section, is also close to the upper bounds for Q. In conclusion, our best estimates of the Q's of Jupiter and Saturn are listed below.

$$\text{For Jupiter} \quad Q \simeq (1-2) \times 10^5$$

$$\text{For Saturn} \quad Q \simeq (6-7) \times 10^4.$$

13. *Further areas for possible investigations.*—In this paper we investigated the stability of commensurabilities under the action of tidal forces. However, no analysis of the formation of these commensurabilities was presented. It is of crucial importance to our arguments to understand this problem in some detail. In particular, we would wish to understand how the amplitude of free libration depends on the conditions of formation of a commensurability. Even more important would be the discovery of some criterion which could tell us the conditions necessary for the formation of a commensurability. At present, these are completely unknown.

Another area for further investigation is the search for stable commensurabilities involving three or more satellites. One possible example of such a case

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is described below. Denoting the elements of Dione, Rhea and Titan by the subscripts 4, 5 and 6 respectively, the following relations may be noted:

$$6n_6 - 5n_5 + 2n_1 = 000{\cdot}081529 \text{ degrees per day} \quad (13)$$

$$\frac{\overline{d}}{dt}(\tilde{\omega}_4 + 2\Omega_3) = 0{\cdot}0815 \pm 0{\cdot}001 \text{ degrees per day} \quad (14).$$

Hence,

$$6n_6 - 5n_5 + 2n_4 - \frac{\overline{d}}{dt}(\tilde{\omega}_4 + 2\Omega_3) = 000{\cdot}0000 \pm 0{\cdot}0001 \text{ degrees per day.}$$

We see that this relation holds to within the observational accuracy of six significant figures. If it is a stable commensurability relation then a direct observation of the tidal phase shift should be possible.

The stable commensurability relations which were discussed in this paper referred only to satellite systems and not to the planetary system. Tidal effects on the planet's orbits are too small to have any significance, even over ages comparable to that of the solar system. However, the stability proof discussed in this paper, would apply equally well to other phenomena which might produce secular changes in the semi-major axis of planets or satellites. In particular during the process of planet formation such forces would undoubtedly have existed in one form or another. It is then possible that the planets might also have been involved in commensurability relations of the types discussed, and that their present distribution of mean motions is at least partially a reflection of these relations.

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