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ON THE ORIGIN OF THE COMMENSURABILITIES AMONGST THE SATELLITES OF SATURN

A. T. Sinclair

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SUMMARY

The hypothesis that the commensurabilities amongst the satellites of Saturn are due to the action of tidal forces is examined. It is shown that this hypothesis provides a satisfactory explanation of the origin of the commensurabilities between Mimas and Tethys, and between Enceladus and Dione. The origin of the commensurability between Titan and Hyperion cannot be explained in this way, but it is possibly the result of a close approach between these satellites.

1. INTRODUCTION

It has been shown by Roy & Ovenden (1954) that the number of occurrences of commensurabilities in mean motions between pairs of satellites in the solar system is greater than would be expected to occur by chance if the mean motions were randomly distributed. Thus we must either suppose that for some reason the satellites were formed preferentially at the commensurabilities, or that some mechanism exists to drive them from their original orbits into the commensurabilities. In this paper we investigate Goldreich's hypothesis (1965) that the satellites were driven into the commensurabilities by the action of tidal forces. We restrict our attention to the satellites of Saturn.

The attraction of one satellite on another is greatest when the satellites are in conjunction. If there is no commensurability relationship between the mean motions, then the point in the orbit of each satellite at which the conjunctions occur will move rapidly around the orbit. The perturbing force of the other satellite will thus act close to any given point of the orbit for a short time only, and a small perturbation in the orbit will result.

However, if a fairly close commensurability relationship exists, then the point at which the conjunctions occur will move more slowly around the orbit. The perturbing force will act close to any given point of the orbit for a longer time, and a larger perturbation will result. We shall refer to this type of motion in which the point of conjunction moves around the orbit as 'circulation'. The closer the commensurability, the slower will be the circulation, and the larger the resulting perturbations.

If the commensurability relationship is very close, then a special type of motion is possible in which the point of conjunction no longer circulates around the orbit, but oscillates (or librates) about a fixed point in the orbit. This type of motion is called 'libration'.

We shall consider situations in which the fixed point is either an apse of one of the satellite orbits, or is related to the position of the nodes of the orbits on the equatorial plane of the planet. In the first case the amplitude of libration can have any value up to 180°, and a libration of amplitude 180° is the borderline case between libration and circulation. This type of libration involves the eccentricities of the orbits, and will be referred to as 'eccentricity-type'.

In the second case the maximum amplitude is 90°. (This is because the ascending and descending nodes are identical points as far as the perturbations of the other satellite are concerned. This is not the case with the two apses, the pericentre and the apocentre.) This type of libration involves the inclinations and will be referred to as 'inclination-type'.

Of the ten satellites of Saturn, six are involved in commensurabilities, and in each case the commensurability is very close and a libration occurs. They are as follows:

(i) the mean motions of Mimas and Tethys are in the ratio 2 : 1, and the conjunction of the satellites librates about the mid-point of the nodes, with amplitude 48°·5;

(ii) the mean motions of Enceladus and Dione are in the ratio 2 : 1, and the conjunction of the satellites librates about the pericentre of Enceladus, with amplitude 1°·5;

(iii) the mean motions of Titan and Hyperion are in the ratio 4 : 3, and the conjunction of the satellites librates about the apocentre of Hyperion, with amplitude 36°.

In the cases where the fixed point depends on the positions of the nodes we shall take as the libration argument twice the angular distance of the conjunction from the fixed point. Thus the maximum amplitude of this argument is 180°, and in the Mimas–Tethys system the actual libration amplitude is 97°.

The tidal force on a satellite is due to the attraction of the bulge that the satellite raises on the planet. In the cases we shall consider the effect of these forces is to increase the orbital angular momentum of the satellite and hence decrease its mean motion. In general, the mean motions of a pair of satellites will be decreased at different rates. Hence in time a commensurability relationship will exist between them. Goldreich's hypothesis is that in one of these situations the effect of the mutual perturbations is to preserve the commensurability. He has shown that libration motion is stable under the action of tidal forces. Thus the commensurability will be preserved if by some means the system enters a libration.

However, according to his analytical treatment of the problem, circulation and libration motions are quite distinct. The system would originally be in circulation-type motion, and as the tidal forces drove it towards the commensurability the circulation would slow down and eventually stop. Thereupon the circulation would simply recommence in the opposite direction, and libration motion would not occur.

Even if we suppose that by some means the libration is entered, say due to a random perturbation by another planet or the Sun, then the amplitude of libration would be close to 180°, so it is necessary to find some mechanism to reduce the amplitude in order to explain the observed values.

Allan (1969) has studied the evolution of the Mimas–Tethys system under the action of tidal forces, assuming it was formed with initial amplitude close to 180° by some means. He shows that the effect of the tidal forces is to reduce the amplitude of libration, and that about $2 \cdot 2 \times 10^8$ yr would be required to reduce it to its present value of 97°.

The Enceladus–Dione commensurability is somewhat different as it is of the eccentricity-type. However, it seems unlikely that tidal forces could have reduced the amplitude to the present very small value of $1^\circ \cdot 5$ within the age of the solar system.

Thus, for the tidal hypothesis to be acceptable we must explain how capture into libration can occur, and why in some cases the present libration amplitude is very small. In this paper we give possible explanations of these points. We show by numerical integrations of the equations of motion that circulation and libration motions are not in fact distinct. They only appear to be so in the analytical treatment due to various approximations that have to be made. For a small range of initial conditions it is possible for the motion to pass from circulation to libration. The capture into libration and the subsequent decrease of amplitude both have the same cause, which is a secular increase of the inclinations or eccentricities (depending on the type of commensurability) due to the combined effects of the tidal forces and the commensurability. In the Mimas–Tethys case it is estimated that the probability of capture into the libration is 0·04.

This method of capture into a large amplitude libration is valid for both types of commensurability, provided the inclinations or eccentricities are large enough. For eccentricity-type commensurabilities, the evolution is somewhat different if the eccentricity is small (the limiting value depends on the masses of the satellites). Then the motion must pass automatically from circulation into a small amplitude libration, with no possibility of entering reverse circulation. This is characterized by the eccentricity being reduced instantaneously to zero. The eccentricity of Enceladus is below the limit for this to occur, so this mechanism provides a reasonable explanation of the origin of this small amplitude libration.

For small inclinations the evolution of the inclination-type commensurabilities is also different from that described above, but we show that entry into a libration will not occur.

Neither of these mechanisms for capture into libration give a satisfactory explanation of the origin of the Titan–Hyperion libration, as the amplitude is too small for the first mechanism to be applicable, and the eccentricity of Hyperion is too large for the second to be applicable. However, we show that an effect of this libration is to prevent close approaches between these satellites, which could otherwise cause drastic changes in the orbit of Hyperion. It would seem that the origin of this libration must be connected with the avoidance of close approaches rather than with the effect of tidal forces.

There is no direct evidence that the satellites of Saturn are affected by tidal forces, as the secular accelerations in longitude that would be necessary to produce the commensurabilities are too small to be detected observationally. Thus we have to make various assumptions about the nature of the tidal forces. These are discussed in the next section.

2. TIDAL FORCES

We adopt the standard formulation of the action of the tidal forces, as has been used by many other authors. See, for example, Jeffreys (1962), MacDonald (1964), Goldreich & Soter (1966).

The rate of change of the mean motion of the satellite due to the action of tidal forces is found to be

$$\frac{dn}{dt} = -\frac{27n^2m}{4Q}\left(\frac{a_0}{a}\right)^5 \tag{1}$$

where m is the ratio of the mass of the satellite to that of the planet, a_0 is the radius of the planet, a is the radius of the satellite's orbit (assumed to be circular). Q is the planetary dissipation function, defined by

$$Q^{-1} = 1/(2\pi E_0)\oint -\frac{dE}{dt}\,dt,$$

where E_0 is the peak energy stored in the tidal distortion, and the integral of $-dE/dt$ is the energy dissipated during one complete cycle of the tide.

In the derivation of equation (1) only the second harmonic term in the tide-raising potential of the satellite is retained. The tidal distortion caused by this potential is assumed to be given by the equilibrium distortion of a homogeneous elastic sphere.

During the process of capture into a libration, n and a vary only slightly, and we shall show that the resulting variation of dn/dt plays no significant part in the capture process. Hence we neglect this variation, and assume a constant value of dn/dt due to the tidal forces. (There are, of course, other variations in dn/dt due to the mutual interactions of the satellites.)

However, equation (1) is of importance in determining the relative values of dn/dt and dn'/dt (where n is the mean motion of the inner satellite, n' that of the outer). For the cases in which we are interested (Mimas–Tethys and Enceladus–Dione) we have $n \doteqdot 2n'$. We put $K = d/dt(2n' - n)$, where we are only considering rates of change due to the tidal forces. The sign of K determines the direction from which the commensurability is approached.

Now dn/dt decreases rapidly with distance from the planet, but for each of the pairs of satellites we consider the outer is about 15 times more massive than the inner. The result is that in each case dn/dt and $2dn'/dt$ are comparable in magnitude.

If we assume that Q has the same value for each satellite of a pair, then we find $K > 0$ for both pairs of satellites.

Very little is known about the structure of Saturn, so it is difficult to make any estimates about possible processes for the dissipation of tidal energy within the planet. However, Goldreich and Soter (1966) have estimated the value of Q that would arise from energy dissipation by boundary layer turbulence at the base of the atmosphere. The value obtained agrees reasonably well with Goldreich's (1965) estimate of a lower bound of Q, obtained from the requirement that the satellites have always been above the planet's surface. Dermott (1968, 1971) modifies their arguments slightly, and shows that the expression for Q for such a tide depends on the amplitude and frequency of the tide. Then the values of Q for each satellite are no longer the same, which has the effect of making $K < 0$ for each of the pairs of satellites in which we are interested.

The sign of K has a considerable effect on the evolution under the action of tidal forces. In his study of the evolution of the Mimas–Tethys libration, Allan (1969) assumes the same value of Q for each satellite, and hence has $K > 0$. He shows that the amplitude of libration is decreasing with time. However, if we assume $K<0$ then the amplitude will increase with time, so ultimately it will reach 180° and the libration will cease.

In this paper we show that with the assumption $K>0$ we can explain the origin of the Mimas–Tethys and Enceladus–Dione librations. However, with $K<0$ we show that for the Enceladus–Dione system capture into libration is not possible. The Mimas–Tethys case is more complicated, but it seems likely that capture into a small amplitude libration would not be possible for $K<0$ (this would be necessary to explain the present amplitude, as the amplitude would increase with time). Thus, if $K<0$ we must conclude that the satellites were formed in the librations, and so tidal evolution of the orbits has been insignificant.

However, Allan (1969) shows that the quantity $am^{-2/13}$ should be a constant for all satellites that have undergone appreciable tidal evolution. He shows that Mimas, Tethys, Enceladus and Dione closely obey this condition, which gives strong grounds for believing that tidal evolution has been significant. If this is the case, then we see that we must have $K>0$.

3. FORMULATION OF THE PROBLEM

We consider the motion of two satellites of a planet in orbits with small inclinations to the equatorial plane. Due to the oblateness of the planet the nodes of these orbits will have appreciable secular motions around the equatorial plane. Thus we take this plane as our reference plane, since to take any other reference plane would lead to large variations in the nodes and inclinations.

We denote the orbital elements of the inner satellite by a, e, i, l, g and Ω, where a is the semi-major axis, e the eccentricity, i the inclination to the equatorial plane, l the mean anomaly, g the argument of the pericentre, and Ω the longitude of the ascending node. We put $\lambda = l+g+\Omega$, $\varpi = g+\Omega$, and we let the ratio of the mass of the inner satellite to that of the planet be m. Similarly, we let the ratio of the mass of the outer satellite to that of the planet be m', and its orbital elements be a', e', i', l', g', Ω', with $\lambda' = l'+g'+\Omega'$, $\varpi' = g'+\Omega'$.

We denote the mean motions in longitude of the satellites by n and n'. These are assumed to include all secular effects in longitude, such as those due to the oblateness of Saturn. Thus they will not satisfy Kepler's law exactly, but for our purposes it will be sufficiently accurate to put $n^2a^3 = n'^2a'^3 = \nu$, say.

Let $\mathbf{r}$, $\mathbf{r}'$ be the position vectors of the satellites relative to the planet, and Δ be the distance between them. Then the disturbing function for the motion of the inner satellite perturbed by the outer is

$$R = \nu m'(1/\Delta - \mathbf{r}.\mathbf{r}'/r'^3)$$

and that for the outer perturbed by the inner is

$$R' = \nu m(1/\Delta - \mathbf{r}.\mathbf{r}'/r^3).$$

In the usual manner, we suppose the disturbing functions to be expanded as sums of terms of the form $C \cos N$, where each coefficient C is a function of a, a', e, e', i and i', and each argument N is a linear combination of l, l', g, g' and $(\Omega-\Omega')$.

The Mimas–Tethys and Enceladus–Dione commensurabilities are both 2 : 1. Thus we assume that $n \doteqdot 2n'$, and define the following arguments:

$$
\begin{aligned}
\theta &= l-2(\lambda-\lambda') \\
\theta' &= l'-(\lambda-\lambda') \\
\phi &= l+g-2(\lambda-\lambda') \\
\phi' &= l'+g'-(\lambda-\lambda').
\end{aligned}
$$

We see that we can express each argument N as a linear combination of these arguments and $(\lambda-\lambda')$. So we may take these as our fundamental arguments instead of l, l', g, g' and $(\Omega-\Omega')$. They are convenient for the study of commensurability motion for the following reasons.

The argument θ becomes indeterminate for $e = 0$. Hence, it must appear in R and R' with a factor e. Similarly, θ', ϕ and ϕ' must appear with factors e', i and i', respectively. The rates of change of θ, θ', ϕ and ϕ' are all approximately $2n'-n$.

The exact rates of change of these arguments differ slightly from $2n'-n$ due to the secular motions of the nodes and apses, caused principally by the oblateness of the planet. For Saturn's satellites the periods of these motions are of the order of a year, so these four arguments are quite distinct. Thus, if any one of these arguments (or a particular linear combination of them) has a very small rate of change (i.e. is a critical argument causing resonance effects) the remaining arguments can be regarded as short-period arguments, as is $(\lambda-\lambda')$. Short-period arguments increase or decrease indefinitely. This is the circulation motion referred to in the introduction. For a critical argument, libration motion is possible.

However, no terms of first power in i and i' appear in R and R', and to the second power of i and i', ϕ and ϕ' only appear in the combinations 2ϕ, $2\phi'$, $\phi+\phi'$ and $\phi-\phi'$; and $\phi-\phi'$ is a short-period argument. These quantities also appear in linear combinations with θ and θ', but it seems very unlikely that such an argument could cause significant resonance effects, as it would have a factor of order ei^2 at least. We shall only consider resonance effects due to the arguments 2ϕ, $2\phi'$, $\phi+\phi'$, θ and θ'. For each of these arguments libration motion is possible. Thus we have five different types of libration to consider at the 2 : 1 commensurability. The first three are of the inclination-type referred to in the introduction, the last two are of the eccentricity-type.

The rates of change of these arguments are

$$
\begin{aligned}
\dot{\theta} &= 2n'-n-\dot{\varpi} \\
\dot{\theta}' &= 2n'-n-\dot{\varpi}' \\
2\dot{\phi} &= 4n'-2n-2\dot{\Omega} \\
2\dot{\phi}' &= 4n'-2n-2\dot{\Omega}' \\
\dot{\phi}+\dot{\phi}' &= 4n'-2n-\dot{\Omega}-\dot{\Omega}'.
\end{aligned}
$$

For the Mimas–Tethys case we have the following approximate numerical values:

$$
\begin{aligned}
&\text{Mimas:} \quad \dot{\varpi} = -\dot{\Omega} = 360\,^\circ/\text{yr} \\
&\text{Tethys:} \quad \dot{\varpi}' = -\dot{\Omega}' = 70\,^\circ/\text{yr}
\end{aligned}
$$

For the Enceladus–Dione case we have

$$
\begin{aligned}
&\text{Enceladus:} \quad \dot{\varpi} = -\dot{\Omega} = 150^\circ/\text{yr} \\
&\text{Dione:} \quad \dot{\varpi}' = -\dot{\Omega}' = 30^\circ/\text{yr}.
\end{aligned}
$$

As explained in Section 2, we assume that the effect of the tidal forces is to steadily decrease $(n-2n')$. Hence we see that in each case these arguments will become critical arguments in the order 2ϕ, $\phi+\phi'$, $2\phi'$, θ', θ. In fact, for the Mimas–Tethys system the argument $\phi+\phi'$ is librating, and for the Enceladus–Dione system the argument θ is librating.

4. MIMAS–TETHYS SYSTEM

4.1 *Analytical treatment of the $\phi+\phi'$ commensurability*

We assume Mimas and Tethys to be in circular orbits, and represent the orbital elements of Mimas by unprimed symbols, and those of Tethys by primed symbols. We put

$$\psi = \phi+\phi' = 4\lambda'-2\lambda-\Omega-\Omega'.$$

As we are only interested in the long-term motion of the satellites, we neglect in the expansions of R and R' all periodic terms except those whose arguments depend on ψ only. The indirect parts of R and R' have no critical terms, and those in a'/Δ are

$$a'/\Delta = -D \sin i \sin i' \cos \psi + O(i^4, i'^4)$$

where $D = \frac{1}{4}\alpha b_{3/2}{}^{(3)}$, and $\alpha = a/a'$. ($b_{3/2}{}^{(3)}$ is one of the Laplace coefficients and is a function of α.) The non-periodic terms in a'/Δ cause secular changes in λ, λ', Ω and Ω' which can all be assumed to be absorbed in the values of n and n'.

We let the secular changes in n and n' due to the tidal forces be $-T$ and $-T'$ respectively. Thus $K = T-2T'$. Then from the Lagrange planetary equations, retaining only the largest terms on the right-hand sides, we obtain the following equations:

$$\frac{dn}{dt} = -T+6n^2\alpha m'D \sin i \sin i' \sin \psi \tag{2}$$

$$\frac{dn'}{dt} = -T'-12n'^2mD \sin i \sin i' \sin \psi \tag{3}$$

$$\frac{di}{dt} = n\alpha m'D \sin i' \sin \psi \tag{4}$$

$$\frac{di'}{dt} = n'mD \sin i \sin \psi \tag{5}$$

$$\frac{d\psi}{dt} = 4n'-2n+\left(n\alpha m'\frac{\sin i'}{\sin i}+n'm\frac{\sin i}{\sin i'}\right) D \cos \psi. \tag{6}$$

Our notation is somewhat different from that used by Allan (1969), but we follow his analytical treatment of these equations. For a first order approximation we neglect terms of order m, m' in equation (6). We obtain

$$\frac{d^2\psi}{dt^2} = 2K-p \sin \psi \tag{7}$$

where $p = (12n^2\alpha m'+48n'^2m) D \sin i \sin i'$. As a first approximation, we assume p to be a constant. Then this is the equation of a simple pendulum with the addition of a constant force K. It has the integral

$$\tfrac{1}{2}\dot{\psi}^2 = 2K\psi+p \cos \psi+E. \tag{8}$$

These equations for ψ are analogous to the equations of motion of a particle in a one-dimensional potential. It can be shown (see Allan (1969)) that for a suitable value of the 'energy' E it is possible for the particle to be trapped in a potential well. Then ψ will oscillate between fixed limits. This corresponds to libration motion. Alternatively, if E is too large for the particle to be trapped, then circulation motion will occur. In this case, if we suppose that initially $\dot{\psi} < 0$, then ψ will decrease continuously until $\dot{\psi} = 0$. Thereupon the motion simply reverses, and ψ increases indefinitely.

So, according to this approximate treatment, circulation and libration motions are distinct. However, we shall show that this is not so if the equations are treated more exactly. Again we follow Allan.

Eliminating $\sin \psi$ from (4) and (7), we obtain

$$\frac{di}{dt} = n\alpha m' D \sin i'(2K - \ddot{\psi})/p.$$

This can be integrated approximately to give the variation in i:

$$\delta i = n\alpha m' D \sin i'(2Kt - \dot{\psi})/p.$$

In a similar manner we can find the variations in n, n' and i', and hence the variation in p, given by

$$\delta p = \frac{\partial p}{\partial i}\,\delta i + \frac{\partial p}{\partial i'}\,\delta i' + \frac{\partial p}{\partial n}\,\delta n + \frac{\partial p}{\partial n'}\,\delta n'. \tag{9}$$

Thus δp contains secular terms and terms depending on $\dot{\psi}$. If we now include the terms of order m and m' in equation (6), we obtain

$$\frac{d^2\psi}{dt^2} = 2K - p \sin \psi - q\dot{\psi} \sin \psi \tag{10}$$

where $q = (n\alpha m' \sin i'/\sin i + n'm \sin i/\sin i')\, D$, and we have neglected terms of second order in the masses. We now suppose p to vary according to equation (9), and q to be a constant. It is easy to verify that the term $q\dot{\psi} \sin \psi$ in equation (10) cancels with the terms in $\dot{\psi}$ arising from the dependence of p on i and i'. Similar terms in $\dot{\psi}$ arise from the dependence of p on n and n', but these are of order mii', $m'ii'$, and are negligible. So the only remaining terms due to the variation of p are the secular terms. Those arising from the dependence of p on n and n' are negligible. Thus we are left with

$$\frac{d^2\psi}{dt^2} = 2K - p \sin \psi - 2qKt \sin \psi, \tag{11}$$

where both p and q are now regarded as constants, and both are positive. Also, as explained in Section 2, we shall assume $K > 0$. Thus we see that the variation of ψ corresponds to motion in a potential well in which the depth of the well is increasing with time.

We consider the system to be in circulation motion with ψ decreasing, just approaching the point at which the circulation changes direction. We see from equation (8) that $\dot{\psi}^2$ is reduced by an amount $8K\pi(=v^2$, say) in each complete revolution. Thus, as it passes the unstable equilibrium point (near $\psi = \pi$) for the final time before the change of direction, we must have $\dot{\psi}^2 < v^2$. We suppose that the actual value of $\dot{\psi}^2$ at this point is u^2. Due to the increase in depth of the potential well the value of $\dot{\psi}^2$ when next at $\psi = \pi$ will be less than u^2 and can be negative if u^2 is small enough. In this case the system has been captured into libration, and the point $\psi = \pi$ is not in fact reached.

If we define u^2 to be the initial value of $\dot{\psi}^2$ at $\psi = \pi$ in order that capture should just occur, then the probability of capture is u^2/v^2 (where we are assuming that all values of $\dot{\psi}^2$ between 0 and v^2 are equally likely. This is equivalent to assuming that the particle has random energy initially). We resort to numerical integrations to determine u^2.

We see that capture into libration is possible because p is increasing with time, primarily due to the increase of i and i' with time. Allan (1969) shows that it is primarily this same effect which causes the subsequent decrease of libration amplitude following the capture.

4.2 *Numerical integrations at the $\phi + \phi'$ commensurability*

The present inclinations of the orbits of Mimas and Tethys are $i = 1°{\cdot}5167$ and $i' = 1°{\cdot}0927$. Allan (1969) shows that when the amplitude of libration was 180° the inclinations would have been $i = 0°{\cdot}4156$ and $i' = 1°{\cdot}0463$. We adopt these values for i and i', and also Allan's estimates of the masses of the satellites, $m = 6{\cdot}776 \times 10^{-8}$ and $m' = 1{\cdot}155 \times 10^{-6}$. Allan uses Goldreich's (1965) estimate of the reciprocal of the planetary dissipation function $Q^{-1} = 1{\cdot}5 \times 10^{-5}$. We take the same value, and hence obtain from equation (1) the values $T = 1{\cdot}003 \times 10^{-13}$, $T' = 4{\cdot}23 \times 10^{-14}$, in units such that $n' = 1$ rad/unit of time. (Thus the unit of time is about 0·300 days.)

The differential equations (2)–(6) were integrated numerically by an Adams–Bashford 10th-order method, starting from the unstable equilibrium point $\psi = \pi - 2K/p$, with i and i' as above. The initial value of n' was 1, and the value of n was chosen to give the desired initial value of $\dot{\psi}$.

The value of α is given by $\alpha = (n'/n)^{2/3}$, and is close to 0·62996 ($=\alpha_0$, say). The value of D is given by $D = 0{\cdot}4099 + 4{\cdot}0954(\alpha - \alpha_0)$, where we have represented the variation of D with α by its first derivative.

The equations were integrated for various initial values of $\dot{\psi}$, in order to find the value u for which capture into libration will just occur, and the value v for which ψ will just perform one more complete revolution (this is approximately given by $(8K\pi)^{1/2}$). They were found to satisfy

$$u = 0{\cdot}2v.$$

Thus the probability of capture into the libration is $(0{\cdot}2)^2 = 0{\cdot}04$.

As a check on the assumptions made in deriving equation (11) from equations (2)–(6), this equation was also integrated numerically, and the results for the variation of ψ with time from the two integrations were compared. They were found to be in good agreement (the maximum discrepancy was about 1°), and gave the same value for the probability of capture into libration.

Further integrations of equations (2)–(6) were performed in which T and T' were allowed to vary according to equation (1). The results were not significantly changed. Also integrations were performed with the values of T and T' first increased and then decreased by a factor of 10. The value of v was increased (or decreased) by a factor of about $10^{1/2}$, but so was the value of u, so the probability of capture into libration was about the same. So we conclude that, provided $K > 0$, the probability of capture does not depend strongly on K.

4.3 2ϕ *commensurability*

As was shown earlier, before encountering the $\phi+\phi'$ commensurability, the Mimas–Tethys system must have passed through the 2ϕ commensurability (assuming $K>0$). Accordingly, the differential equations for this case were integrated in a similar manner to that described above. The probability of capture into the libration was found to be 0·07.

4.4 *Summary*

We have shown that for suitable initial conditions it is possible for capture into libration to occur at the 2ϕ and $\phi+\phi'$ commensurabilities due to the action of tidal forces. The ranges of initial conditions for which capture will occur are small, so the probabilities of capture are small, 0·07 and 0·04 respectively. The probabilities are not critically dependent on the magnitudes of the tidal forces.

The Mimas–Tethys system is in libration at the $\phi+\phi'$ commensurability. We see that the tidal theory provides a possible explanation of the origin of this libration, although capture into such a libration would be a rare event. This explanation of the origin of the libration is only valid if $K>0$.

5. ENCELADUS–DIONE SYSTEM

5.1 *Effect of tidal forces on libration*

We shall first consider the motion of this system under the action of the arguments θ, θ'. We assume the satellites to be moving in the equatorial plane of Saturn, and represent the elements of Enceladus by unprimed symbols, and those of Dione by primed symbols. The differential equations are

$$\frac{dn}{dt} = -T+3n^2\alpha m'Ae\sin\theta-3n^2\alpha m'Be'\sin\theta'$$

$$\frac{dn'}{dt} = -T'-6n'^2mAe\sin\theta+6n'^2mCe'\sin\theta'$$

$$\frac{de}{dt} = n\alpha m'A\sin\theta$$

$$\frac{de'}{dt} = -n'mC\sin\theta'$$

$$\frac{d\theta}{dt} = 2n'-n+f(a, a')+n\alpha m'A\cos\theta/e$$

$$\frac{d\theta'}{dt} = 2n'-n-n'mC\cos\theta'/e'.$$

To obtain the differential equations for the action of one argument alone we simply neglect all terms depending on the other argument. Again we represent the effects of the tidal forces by the terms T, T'. The term f in $\dot{\theta}$ is the difference of the secular rates of θ and θ' due to all effects other than resonance. The secular rate of θ' is assumed to be absorbed in the values of n and n'. The value of f is approximately $-120°$/yr. This term is only of importance when considering the combined effects of the two arguments. If the values of n and n' are such that $\dot{\theta}$

is small, then θ' will be a short-period argument and can be ignored. In this case we shall neglect the term f, assuming it to be absorbed in n and n'.

A, B and C are the following functions of the Laplace coefficients, with numerical values for $\alpha = 0{\cdot}62\,996$. We see that in this case $C = B$, so we shall replace C by B subsequently. D denotes differential w.r.t. α.

$$A = \tfrac{1}{2}(4+\alpha\mathrm{D})\, b_{1/2}{}^{(2)} = 1{\cdot}1905$$

$$B = \tfrac{1}{2}(3+\alpha\mathrm{D})\, b_{1/2}{}^{(1)} - 2\alpha = 0{\cdot}4284$$

$$C = \tfrac{1}{2}(3+\alpha\mathrm{D})\, b_{1/2}{}^{(1)} - (2\alpha^2)^{-1} = 0{\cdot}4284$$

We first of all consider the motion under the action of the argument θ alone. Thus we neglect f. We obtain

$$\frac{d^2\theta}{dt^2} = K - p\sin\theta - \tfrac{1}{2}q^2\sin 2\theta - q\dot{\theta}\sin\theta$$

where $K = T - 2T'$, $p = (3n^2\alpha m' + 12n'^2 m)\,Ae$, $q = n\alpha m'A/e$. As in the Mimas–Tethys case, we shall neglect the $\dot{\theta}$ term initially, and we shall assume p and q to be constants. Although the term in $\sin 2\theta$ is of order m'^2 we cannot neglect it as it can be significant for small values of e. However even with this term included the equation still possesses an integral, and it can be shown by similar arguments to those used for the Mimas–Tethys case that again two distinct types of motion are possible, circulation and libration.

In a similar manner, we can show that circulation and libration motions are possible under the action of the θ' argument alone.

For a sufficiently large value of e or e', the θ and θ' systems behave in a similar manner to the Mimas–Tethys system. There is a small probability of capture into a libration of amplitude close to 180°. The capture results from the increase in depth of the potential well due to the increase of e or e' with time. Again this capture mechanism is only valid for $K > 0$.

However, in the Enceladus–Dione system where the eccentricities are small it is possible for the perturbations in e and e' due to the commensurability to be of the same order as e and e'. The analytical approach used earlier breaks down, and other methods are needed. These are described in the next two sections.

5.2 θ' *commensurability*

Commensurability motion in the absence of tidal forces has been extensively studied, and we adopt one of the methods used to give a qualitative description of the motion with tidal forces included. The method we adopt is similar to that used by Message (1966) and Schubart (1964) to study the motion of asteroids close to commensurabilities with Jupiter.

We require a formulation of the equations of motion in canonical variables, with one Hamiltonian to describe the motion of both bodies. Such a formulation is given by Sinclair (1970), p. 338, from where we derive the following results for the motion at the θ' commensurability in the absence of tidal forces.

We put $L = \beta(\nu a)^{1/2}$, $L' = \beta(\nu a')^{1/2}$, $G = L(1-e^2)^{1/2}$, $G' = L'(1-e'^2)^{1/2}$, where β and β' are the masses of the satellites, and $\nu = n^2a^3 = n'^2a'^3$. (N.B. We are neglecting terms of second order in the masses.) We shall put $m = \beta/M$, $m' = \beta'/M$, where M is the mass of Saturn.

We take θ, θ' and $(\lambda-\lambda')$ as our coordinates. The conjugate momenta are respectively

$$\Theta = L-G; \quad \Theta' = L'-G'; \quad \Phi = 2L-G+L'-G'.$$

We also have the integral $G+G' = \Psi$, constant. The Hamiltonian is

$$F = -\tfrac{1}{2}\nu^2\left(\frac{\beta^3}{L^2}+\frac{\beta'^3}{L'^2}\right)-\frac{\nu\beta\beta'}{Ma'}R$$

where

$$R = a'(1/\Delta-\mathbf{r}.\mathbf{r}'/r'^3)$$

and Δ is the distance between the satellites, $\mathbf{r}$ and $\mathbf{r}'$ are their position vectors relative to the centre of mass of Saturn and Enceladus. This origin is necessary in order to obtain one Hamiltonian for the motion of both satellites. The elliptical elements of the satellites are also referred to this origin. However, for our purposes it is sufficiently accurate to assume this origin to be at the centre of Saturn.

We assume F to be expressed as a function of Θ, Θ', Φ, Ψ, θ, θ' and $(\lambda-\lambda')$. We neglect terms depending on $(\lambda-\lambda')$ as it is a short-period argument. Thus we obtain the integral Φ = constant. Similarly, we shall assume θ to be a short-period argument, so we obtain Θ = constant. We shall take $e = 0$, giving $\Theta = 0$. Since F is independent of time we also have the integral F = constant.

The masses of the satellites are very small compared to that of Saturn (about 10^{-6}) and as a result the perturbations due to the commensurability are small (about 10^{-3}). So if e' is small initially it will always remain small. Thus we can expand F in powers of e', or more conveniently in powers of $\Theta' = O(e'^2)$. To the second power of Θ' this gives

$$F = -X\Theta'-\tfrac{1}{2}\nu^2(3\beta^3\Phi^{-4}+12\beta'^3(\Phi-\Psi)^{-4})\,\Theta'^2-\frac{\nu\beta\beta' R}{Ma'} \tag{12}$$

where

$$X = \nu^2(\beta^3\Phi^{-3}+2\beta'^3(\Phi-\Psi)^{-3}).$$

We have dropped a constant term in F. Now $\Phi = \beta(\nu a)^{1/2}+O(\beta' e'^2)$ and $\Phi-\Psi = -\beta'(\nu a')^{1/2}+O(\beta' e'^2)$. Hence we obtain $X = n-2n'+O(e'^2)$.

For given values of Φ and Ψ the equation F = constant gives us a relationship between θ' and Θ'. By plotting the curve of Θ' against θ' for various values of the constant F, we can describe all types of motion possible for the particular values of Φ and Ψ. By doing this for a range of values of Φ and Ψ we could describe all possible types of motion at the commensurability, but to do this for two parameters Φ and Ψ would be very inconvenient.

However, for close commensurability motion X is small, and small changes in Φ and Ψ will produce changes in X which are of the same order as X. These changes in Φ and Ψ would have a small effect on the coefficient of Θ'^2. Similarly, the effect on the coefficients of Θ' and θ' in the expansion of R would be small. Thus by varying X, and giving all other coefficients in equation (12) fixed values, we can describe all possible types of motion close to the exact commensurability, but we have only one parameter to vary. Now Θ' depends on n' and e', but the principal variation of Θ' is due to e'. Thus we shall regard Θ' as a function of e' only, ignoring its variation with n'.

R appears in equation (12) with a small factor $\beta\beta'$, so we need only expand it to the first power of e', giving $R = Be' \cos\theta'$ (B as defined earlier). Equation (12) then gives

$$\mu e'^4 + Ze'^2 + \tfrac{2}{3}mBe'\cos\theta' = H, \quad \text{constant}, \tag{13}$$

where $\mu = 1 + m'/(4m\alpha^2)$, $\alpha = a/a'$, $Z = X/(3n')$ and $H = -2a'F/(3\nu\beta')$. Numerical values for the Enceladus–Dione system are $\alpha = 0{\cdot}63$, $m = 1{\cdot}27\times10^{-7}$, $m' = 1{\cdot}82\times10^{-6}$, $B = 0{\cdot}4284$, $\mu = 10{\cdot}03$.

We put $x = e'\cos\theta'$ and $y = e'\sin\theta'$. In Fig. 1 we have plotted curves of y against x, with $Z = 0{\cdot}0$, for various values of H. The curves in Figs 2 and 3 have $Z = -3{\cdot}66\times10^{-5}$ and $-14{\cdot}7\times10^{-5}$ respectively, and are again plotted for various values of H. Only the upper half plane is shown as the curves are symmetrical about $y = 0$. These small values of Z imply small values of $n-2n'$, and thus correspond to close commensurability cases. The curves enclosing the origin correspond to circulation of θ'. Those not enclosing the origin correspond to libration. The variation in radial distance on a given curve gives the variation of e'. Thus these curves describe the variation of e' with θ', but not the variation of e' or θ' with time. The arrows on the curves show the direction of motion.

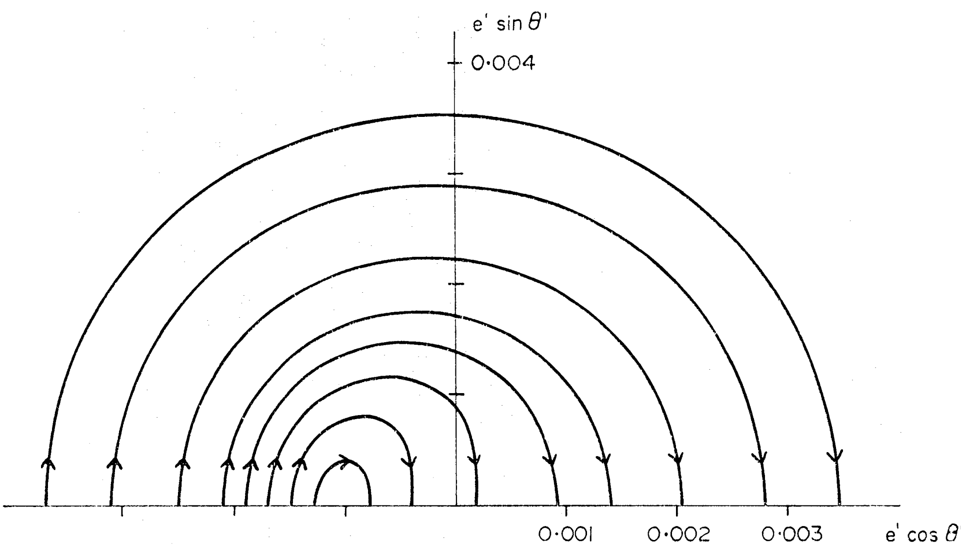

FIG. 1. *θ' commensurability.* $Z = 0{\cdot}0$.

In Fig. 1 only clockwise circulation and clockwise libration occur. In Fig. 2 a bifurcation has just occurred near $x = 0{\cdot}0007$, and around this a small region of anti-clockwise libration exists. In Fig. 3 this region is much larger, and now encloses the origin, so in fact most of it is a region of anti-clockwise circulation.

These diagrams describe the types of motion possible close to the commensurability in the absence of tidal forces. Now the time taken for the system to move around one of the curves is small compared with the time required for significant tidal evolution to occur. Thus, even in the presence of tidal forces, these curves describe the behaviour of the system for many revolutions around the curves. We can think of the effect of the tidal forces as slowly moving the system from one curve to another. Thus we have to find how the parameters H and Z defining the curve vary with time due to the tidal forces. Now

$$Z = \frac{1}{3n'}\left[n - 2n' - e'^2\left(\frac{3m'n\alpha^{-1/2}}{2m} + 6n'\right)\right] + O(e'^4).$$

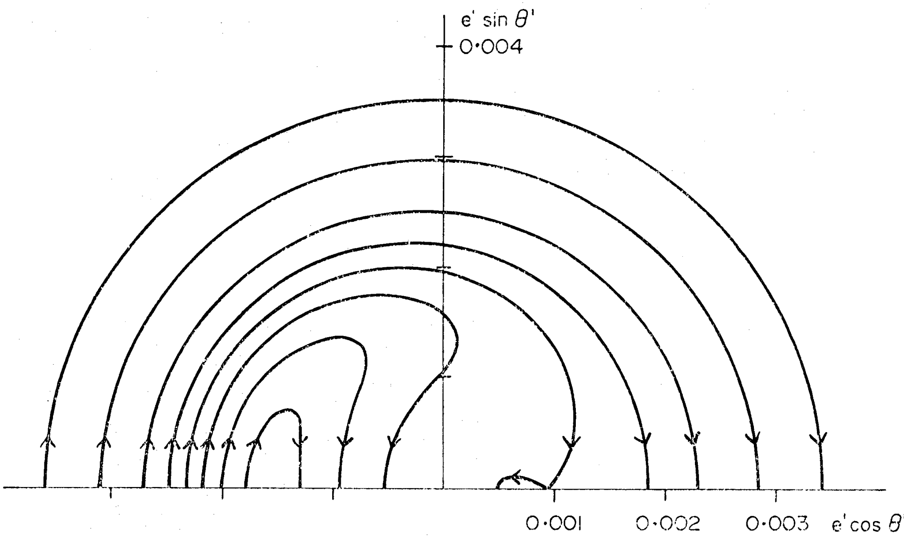

FIG. 2. *θ′ commensurability.* $Z = -3{\cdot}66 \times 10^{-5}$.

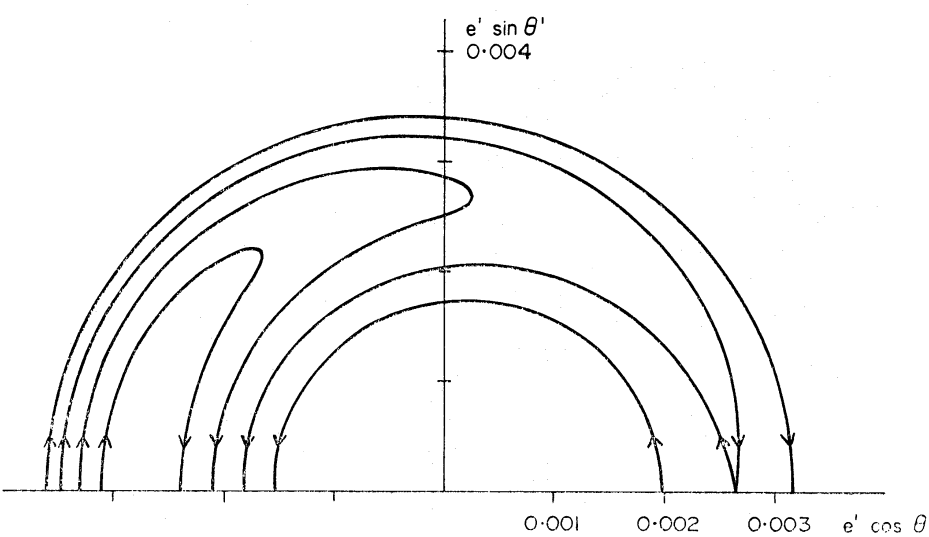

FIG. 3. *θ′ commensurability.* $Z = -14{\cdot}7 \times 10^{-5}$.

We differentiate this equation, allowing only the terms $(n-2n')$ and e'^2 to vary. (The derivatives of the other terms will have a factor of $(n-2n')$ or e'^2, and will be small.) Substituting the expressions for $\dot{n}$, $\dot{n}'$ and $\dot{e}'$ given in Section 5.1 (but neglecting terms in these expressions involving θ, as this is a short-period argument) we obtain

$$\frac{dZ}{dt} = -\frac{K}{3n'}$$

where $K = T-2T'$. As before, we assume $K > 0$. Thus we see that the effect of the tidal forces is to steadily decrease Z.

Since Z varies with time the Hamiltonian is no longer constant. Thus we have $\dot{H} = e'^2\dot{Z}$. Now e'^2 will have periodic variations. These will have a negligible effect on H and we can thus replace e' by its mean value, e'_{m} say, giving

$$\frac{dH}{dt} = e'_{\mathrm{m}}{}^2 \frac{dZ}{dt}. \tag{14}$$

For circulation motion it is reasonable to take e_m' as the value of e' at $\theta' = \pi/2$, since $\dot{e}'$ varies as $\sin\theta'$. Thus the perturbations in e' will vary as $\cos\theta'$, and will be zero at $\theta' = \pi/2$. Hence we take e_m' as the intercept of the circulation curve with the y-axis, and it is given by

$$\mu e'_m{}^4 + Ze'_m{}^2 = H.$$

The rate of change of e'_m is thus given by

$$(4\mu e'_m{}^3 + 2Ze'_m)\frac{de'_m}{dt} + e'_m{}^2\frac{dZ}{dt} = \frac{dH}{dt}. \qquad (15)$$

Comparison of equations (14) and (15) shows that $\dot{e}'_m = 0$. Thus for a given system in circulation motion, the intercept of its curve on the y-axis remains constant as Z and H vary.

To summarize, the effect of the tidal forces is to make Z decrease uniformly with time. Thus Figs 1, 2 and 3 describe all possible types of motion at successive times in the evolution under the action of tidal forces. Whilst in circulation motion the evolution of any particular system is such that the intercept on the y-axis remains constant, and this intercept gives us e'_m, the mean value of e'.

For large values of Z (i.e. well before the commensurability is encountered) all the curves will be concentric circles, centre the origin. We see that in Fig. 1 all systems with e'_m less than about 0·001 have changed from circulations into librations. This occurs because the centre of the curve moves away from the origin until the curve passes through the origin (at this point $e' = 0$). There is no possibility of these systems entering reverse circulation.

Similarly we see in Fig. 2 that all systems with e'_m less than about 0·0019 have changed into librations. For systems with e'_m larger than this value entry into the clockwise libration is no longer automatic, as now the possibility of entry into the small anti-clockwise region exists. We see from Fig. 3 that this region ultimately becomes a region of reverse circulation.

In Fig. 3 we have a similar situation to that considered in the Mimas–Tethys case. For a system in clockwise circulation two possibilities exist; to change into clockwise libration or to change into anti-clockwise circulation.

At present in the Enceladus–Dione system θ' is in anti-clockwise circulation, with $e' = 0{\cdot}0022$. Fig. 3 is such that the limiting anti-clockwise circulation curve corresponds approximately to this eccentricity. Its corresponding clockwise circulation curve has $e'_m = 0{\cdot}0032$. Thus we would suggest that the value of e' before the θ' commensurability was encountered was 0·0032, and that the effect of the commensurability was to reverse the direction of circulation, and to reduce e' to its present value.

We can estimate the probability of this event by the same method we used for the Mimas–Tethys case. We assume the same form for the tidal forces, and the same value of Q, giving $T = 5{\cdot}41\times10^{-14}$ and $T' = 1{\cdot}93\times10^{-14}$, in units such that $n' = 1$ rad/unit of time. (Thus the unit of time is about 0·436 days.) The probability of this event (i.e. that capture into libration does not occur) is found to be 0·81.

Using these values of T and T', we find that the change in the direction of circulation occurred about $1{\cdot}4\times10^8$ yr ago, and that the time required for evolution from Fig. 1 to Fig. 2 was $8{\cdot}5\times10^6$ yr, and from Fig. 2 to Fig. 3 was $2{\cdot}5\times10^7$ yr.

We see that the motion will enter libration automatically (i.e. with no possibility of reverse circulation) if e'_m is less than some limiting value, e'^* say. From equation (13) we obtain the following expression for this quantity:

$$e'^* = 2{\cdot}5(mB/12\mu)^{1/3}.$$

For the Enceladus–Dione system this expression gives $e'^* = 0{\cdot}0019$.

According to our simplified model, the tidal forces have no secular effects on the eccentricity when the system is in circulation motion. Goldreich (1963) suggests that in fact the effect would be to reduce the eccentricity. However, the change over the time we are considering of about 10^7 yr would probably be negligible, and in any case it would not affect our qualitative description of the evolution.

5.3 *θ commensurability*

After passing through the θ' commensurability the Enceladus–Dione system next encounters the θ commensurability. To study this commensurability we neglect terms in θ', and assume $e' = 0$. We obtain the expansion of the Hamiltonian in the same manner as before, giving

$$\eta e^4 + Ze^2 - m'\alpha Ae \cos\theta = H$$

where

$$\eta = \frac{3}{8}\left(1 + \frac{4\alpha^2 m}{m'}\right)$$

and

$$Z = \frac{1}{2n}\left[n - 2n' - e^2\left(\frac{3}{2}n + \frac{6mn'\alpha^{1/2}}{m'}\right)\right] + O(e^4).$$

The x–y curves will be similar in shape to those for the θ' case, except that they will be reflected in the y-axis. However, the scale will be different. The expression for the limiting eccentricity in order that the motion should pass automatically from circulation into libration (with no possibility of reverse circulation) is found to be

$$e^* = 2{\cdot}5(m'\alpha A/8\eta)^{1/3}.$$

For the Enceladus–Dione system this gives $e^* = 0{\cdot}019$. At the present time in this system θ is in libration, with $e = 0{\cdot}0044$. Thus we conclude that this system must have passed automatically into the libration, with no possibility of entering reverse circulation.

The present amplitude of libration is $1^\circ{\cdot}5$. A more detailed analysis than this would be necessary to find out how the amplitude varies with time, and thus to determine the eccentricity of the orbit of Enceladus before the commensurability was encountered.

In this discussion we have assumed $K > 0$. If $K < 0$ then the commensurabilities would be encountered in the reverse order, so the θ commensurability would be encountered first, but in the opposite direction to that considered above. It is clear from examination of the diagrams, assuming them to be encountered in the order Fig. 3, Fig. 2, Fig. 1, that automatic entry into a small amplitude libration is not possible. Also capture into a large amplitude libration is not possible, since the depth of the potential well will be decreasing with time. Thus the observed libration cannot be explained by the tidal theory with the assumption $K < 0$.

5.4 *Inclination commensurabilities*

Before reaching the θ' and θ commensurabilities, the Enceladus–Dione system must have passed through the inclination commensurabilities, ϕ, $\phi+\phi'$ and ϕ'. The present inclinations of the satellites are very small (both about $1'\cdot 4$), so we must see if it is possible in these cases for the libration to be entered automatically for small inclinations.

The ϕ commensurability has been investigated in the same manner as for the θ' and θ commensurabilities, and it is found that automatic entry into the libration will occur if the initial inclination is less than i^*, where

$$\sin(i^*/2) = (2m'\alpha E/\xi)^{1/2}$$

with $E = \frac{1}{2}\alpha b_{3/2}{}^{(3)}$ and $\xi = 6(1+4m\alpha^2/m')$. For the Enceladus–Dione system this expression gives $i^* = 3'\cdot 6$, so the libration should have been entered automatically. We must explain why this did not occur.

The differential equations for the ϕ commensurability are

$$\frac{dn}{dt} = -T - 6n^2\alpha m' E\sigma^2 \sin 2\phi$$

$$\frac{dn'}{dt} = -T' + 12n'^2 m E\sigma^2 \sin 2\phi$$

$$\frac{d\sigma}{dt} = -\tfrac{1}{2}n\alpha m' E\sigma \sin 2\phi$$

$$\frac{d\phi}{dt} = 2n' - n - \tfrac{1}{2}n\alpha m' E \cos 2\phi$$

where $\sigma = \sin i/2$. Hence we obtain

$$\frac{d^2\phi}{dt^2} = K + \tfrac{1}{2}\gamma^2 \sin 2\phi + \text{term in } \dot{\phi}$$

where $K = T - 2T'$ and $\gamma^2 = (12n^2\alpha m' + 48n'^2 m)\, E\sigma^2$. As before, if we neglect the term in $\dot{\phi}$ we have the simple pendulum equation, with the addition of the constant force K. Provided $K \ll \gamma^2$ the only effect of this term is to move the equilibrium point slightly, and a stable libration is possible. However, if we take $i = 1'\cdot 4$, we find $K = 1\cdot 55 \times 10^{-14}$ and $\gamma^2 = 2\cdot 07 \times 10^{-12}$ (in units such that $n' = 1$). So in this case γ^2 is not very much larger than K.

For a libration to be entered automatically the inclination must at some time fall to zero. Then $\gamma^2 = 0$, so the K term will dominate the motion. Thus a libration will not be formed. Similar arguments apply to the ϕ' case. The $\phi+\phi'$ case is rather more complicated as it involves both inclinations. However, if both are small, then γ^2 will be small in this case also.

Essentially we are saying that the shapes of the curves in the x–y plane change significantly due to the tidal forces in the period of the motion around the curve. This is not the case for the eccentricity commensurabilities, where the frequency becomes infinite at the origin. In this case the curves in the x–y plane provide a good description of the motion for many revolutions around the curves.

Finally we note that assuming $K = 1\cdot 55 \times 10^{-14}$, the ϕ resonance would have been encountered by the Enceladus–Dione system $4\cdot 3 \times 10^8$ yr ago.

5·5 *Summary*

We have shown that under the action of tidal forces with the assumption $K > 0$, the Enceladus–Dione system would first of all encounter the ϕ, $\phi + \phi'$ and ϕ' commensurabilities. As the inclinations are small it would pass through these, and next encounter the θ' commensurability. At this time the value of e' would be about 0·0032, which would be too large for automatic entry into libration. The probability of capture into the libration would be 0·19. In fact the system was not captured and the evolution continued until the θ commensurability was encountered. The value of e was sufficiently small for automatic entry into libration, and this is the present state of the system.

6. TITAN–HYPERION SYSTEM

This system is in libration at the 4 : 3 commensurability. The libration involves the eccentricity of Hyperion. Thus the critical argument is $\chi = l' - 3(\lambda - \lambda')$, where the unprimed symbols denote the orbit of Titan, and the primed symbols that of Hyperion. The libration is about $\chi = 180°$ with amplitude 36°. The present value of e' is 0·104.

Automatic entry into this libration would occur for $e' < 0{\cdot}068$. Thus we see that if either of the methods we have described were responsible for the origin of this libration, then considerable evolution must have occurred since capture in order to change e' and the amplitude to their present values. Goldreich (1965) shows that the tidal forces in this system are probably too small to have caused significant evolution. We shall show that the perturbations of Hyperion by Titan would completely obliterate any effects of tidal forces, so whatever the origin of this libration may be, it is not due to the action of tidal forces.

We note that for a 4 : 3 commensurability, $\alpha = 0{\cdot}825$. Thus the orbits of the satellites are quite close together, so if a conjunction occurred when Titan was at apocentre and Hyperion at pericentre then the satellites could be very close to each other. In fact the distance of closest approach would be $0{\cdot}05a'$ (N.B. $e = 0{\cdot}03$). However this situation cannot in fact occur. The libration ensures that Hyperion is always close to apocentre when conjunctions occur.

In our descriptions of the evolution of orbits over long periods of time we have been neglecting all short-period perturbations. However, if a close approach occurs, then these perturbations can become large, and the long-period theory cannot be used to predict the motion after the close approach.

To find how close an approach must be for a drastic change in the orbit to occur, we examine the asteroid orbits. None of the asteroids (except Hidalgo, which has a comet-like orbit) can approach Jupiter to within 1 *AU*, so that it would seem that any body with a close approach distance less than this would be liable to have its orbit drastically changed. Marsden (1970) has examined the orbits of many short-period comets, and his results show that any body passing within 0·5 *AU* of Jupiter is very likely to have its orbit drastically changed. At 1 *AU* the attraction of Jupiter is 0·018 times that of the Sun. At 0·5 *AU* it is 0·088 times that of the Sun.

Now the mass of Titan is $2{\cdot}4 \times 10^{-4}$ times that of Saturn. Thus we find that if a close approach of Hyperion to Titan as described above of $0{\cdot}05a'$ could occur, then the attraction of Titan would be 0·077 times that of Saturn. This is well into the danger region, and a drastic change in the orbit of Hyperion would be likely.

However, the closest approach that is possible occurs at a conjunction when $\chi = 180° \pm 36°$, with Titan at apocentre. This distance apart is then about $0{\cdot}14a'$, at which point the attraction of Titan is 0·015 times that of Saturn. Thus we see that the system as it is at present is just out of the danger region. We see that the system could not have been driven into the commensurability by tidal forces, as the orbit before the libration was established would suffer frequent close approaches. The resulting drastic changes in the orbit of Hyperion would obliterate the steady effect of the tidal forces. (We note that there is no problem with close approaches in the other systems we have considered, as the masses and eccentricities are too small.)

It would seem that the origin of this libration must be connected with the avoidance of close approaches. Perhaps Hyperion was one of many bodies formed in this region, and now only it remains, as it happened to be formed in an orbit which avoided close approaches to Titan. Alternatively, perhaps the libration originated following a close approach between the satellites. In this case some other effect, such as a perturbation by another planet, would be necessary to reduce either the amplitude of libration or the eccentricity of Hyperion before another close approach occurred.

7. CONCLUSIONS

We have shown that, provided the tidal forces are large enough to have caused significant orbital evolution, and provided they act in such a way as to drive the inner satellite of a pair away from Saturn relatively faster than the outer, then this mechanism provides a reasonable explanation of the origin of the commensurability between Mimas and Tethys, and of that between Enceladus and Dione. The commensurability between Titan and Hyperion could not have been formed in this manner, but was possibly the result of a close approach between the satellites.

Royal Greenwich Observatory, Herstmonceux Castle, Hailsham, Sussex.

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