win, London, 1979), Prob. 8.30, p. 241.

[3]H. Lamb, *Dynamics* (Cambridge U. P., Cambridge, 1926), Ex. XII (4), p. 149; R. A. Becker, *Introduction to Theoretical Mechanics* (McGraw-Hill, New York, 1954), Prob. 8.9, p. 188; R. B. Leighton, *Feynman Lectures on Physics Exercises* (Addison-Wesley, Reading, MA, 1964), Prob. 14.9; and A. P. French, *Newtonian Mechanics* (Norton, New York, 1971), Prob. 10.17, p. 416. The phraseology is Lamb's, with length $2a$ replaced by L. We further use μ to denote mass per unit length, and assume that the radius of the peg is small.

[4]The equivalent of Eq. (5) appears in N. Feather, *Vibrations and Waves* (Penguin, Harmondsworth, 1964), p. 25, Eq. (17), as part of a derivation of the speed of propagation of a wave pulse along a stretched string.

[5]One is reminded of the well-known problem of a skier sliding down a large spherical snowball. See, for example, F. W. Sears, M. W. Zemansky, and H. D. Young, *University Physics* (Addison-Wesley, Reading, MA, 1982), 6th ed., Prob. 6-42, p. 141.

Stellar structure and the art of building boats

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Acton and Squire's [*Solving Equations with Physical Understanding* (Hilger, Bristol, 1985)] "trial function" method of handling boundary value problems is used to find approximate solutions to the equations of stellar structure; the calculations involved are as simple to carry out as the usual order-of-magnitude estimates, but are correct to a few tens of a percent. The method is also used to investigate why an order-of-magnitude estimate of solar luminosity can be wrong by up to three orders of magnitude, while a similar estimate of the central temperature can be closer to acceptable values.

I. INTRODUCTION

The theory of stellar structure has a well-earned reputation for being a difficult subject; analytical analyses are often obscure with little obvious relevance to real stars, while numerical models of realistic objects are so complicated and so full of parameters that the physical basis on which they are built often disappears from sight.

This is rather unfortunate, since there is much good elementary physics in stellar structure, and the basic equations are rather simple.

Four general equations [(1)–(4) below] determine the equilibrium state of a star, modeled in the simplest case as a spherical gaseous object producing energy in some central portion. The first two are an expression of mechanical equilibrium—gaseous pressure balances gravitation everywhere. The third equation describes the thermal equilibrium, and the fourth is an expression of energy transfer through the stellar volume (the relation given here is a particular example for the case where the energy is transferred only by radiation).

$$\frac{dP}{dr} = -\rho(GM/r^2), \tag{1}$$

$$\frac{dM}{dr} = 4\pi r^2\rho, \tag{2}$$

$$\frac{dL}{dr} = 4\pi r^2\rho\epsilon, \tag{3}$$

$$\frac{dT}{dr} = -\frac{3}{4\alpha c}\frac{\chi\rho}{T^3}\frac{L}{4\pi r^2}, \tag{4}$$

where P is the pressure at distance r from the stellar center; M is the mass within radius r; L is the radiation crossing the surface at radius r; T is the temperature at r; ρ is the density at radius r; ϵ is the rate at which energy is generated per unit mass at radius r; χ is the opacity of the stellar material at radius r; α is Stefan's radiation density constant; and c is the speed of light.

In general, ϵ and χ are functions of density and temperature, and, as a consequence, these simple equations have no proper analytical solutions for the usual case where energy is being generated in some limited section of the stellar core. Numerical modeling, via a step-by-step integration of the original equations, is an essential feature of work in this domain; the numerical procedures are not simple, however, since the stellar parameters run over many orders of magnitude from the center to the outside so that, without special care, one's computer can easily produce significant quantities of rubbish (albeit perfectly plausible rubbish). Numerical modeling of stellar structure can be a good subject for a course in numerical methods but has little place in a physics curriculum.

One's first exposure to the theory of stellar structure is usually via a simple artifice: All quantities that are functions of r are replaced by "typical values" (whatever that may mean), and derivatives are replaced by ratios of typical values. In this way, the original differential equations become a set of simple algebraic identities that may be solved to give "typical" numerical values.

It is rather surprising that such calculations do produce starlike objects; many texts (Ref. 1 is an early example, which has hardly been improved upon for pedagogical clarity) begin by showing that the central temperature of the Sun must be about 20 million deg and predict its luminosity to within a factor 1000 of the observed value. This, in some sense, "sets the scene" (or fixes the ball park, depending on the continent); the technique has been taken to its logical conclusion by Greenstein,[2] who included nuclear reaction processes in this type of analysis. It is often claimed that the technique offers an order-of-magnitude estimate of stellar structure; indeed, Greenstein produced what he referred to as an "order-of-magnitude analytic theory."

In my experience, more thoughtful students tend to be puzzled rather than illuminated by these calculations. They are, of course, suitably grateful for the ease with which the subject is handled; they are, however, perturbed by the apparently light-hearted way in which quantities that vary over many orders of magnitude are replaced by "representative values" (can the average of infinity and zero really represent anything but an arbitrary value?), especially when they learn soon after just how nonlinear the whole business is.

In one sense, these "order-of-magnitude" calculations are just dimensional analysis masquerading under another name (this is not always explicitly stated in textbooks, but one good exception is Rose[3]); in this form, the technique is perfectly justifiable and can yield valuable insights into the functional dependence of stellar parameters.

However, dimensional analysis as such cannot yield the values of constants, and is, therefore, intrinsically ill suited to numerical evaluation. To do this, one has to estimate the order of magnitude of the values of the constants and to assume that single point "typical values" are to within an order of magnitude representative of the variables. Under certain circumstances and for certain objects, this procedure can be justified, and was used in 1966 by Salpeter[4] to discuss the elements of stellar structure; more recently, it was applied with considerable elegance by Weisskopf[5] to show in an elementary way how basic physics determines the evolutionary history of a star, and by Nauenberg and Weisskopf[6] to derive an *a priori* estimate for the solar luminosity and for the solar radius from a calculation of nuclear reaction rates (of which a simplified version appears in Greenstein[2]); a textbook based on this approach has been written by Sexl and Sexl,[7] while various astrophysical examples have been discussed by, for example, Carr and Rees,[8] Dyson,[9] and Celnikier.[10,11]

The most elementary of these calculations suffers from an important defect. To obtain an answer, it is necessary to assume either the stellar radius or the mean density; one obtains a central temperature (which is not an observable and can only be compared to a numerical model) and a luminosity, which is observable, but can turn out to be wrong by up to three orders of magnitude (depending on just how one rounds the numbers). It is in many ways unfortunate that many of these analyses use an *a priori* value for the stellar radius since a fundamental tenet of stellar astrophysics with very broad implications is that the structure of a star is uniquely determined by its mass and chemical composition. Nauenberg and Weisskopf,[6] and Greenstein,[2] have alleviated this problem by using nuclear reaction rates to obtain interesting analytical expressions for observable stellar parameters as explicit functions of mass but even their analyses are flawed since they involve the volume of the star participating in the nuclear reactions and this cannot emerge from a model based on the notion of "typical values" of quantities that vary over many orders of magnitude.

A plausible order-of-magnitude analysis of stellar structure in fact requires considerable skill, first in deciding how to calculate the orders of magnitude involved, and then in choosing the variables whose orders of magnitude will be calculated. It is particularly instructive to compare Nauenberg and Weisskopf's article[6] with that of Greenstein.[2] The former contains a plausible "fudge factor" to take account of the nonuniform density distribution; assuming that radiation is transferred through the Sun by electron scattering, they derive a solar luminosity and radius, which are, respectively, only twice and $\frac{1}{3}$ of the observed values. Greenstein, taking a uniform model, has no arbitrary degrees of freedom for the density distribution, but introduces instead his own plausible fudge factor in the form of an opacity exceeding the electron scattering opacity by an arbitrary amount; in this way he, too, obtains some degree of agreement.

In point of fact, the opacity used by Greenstein is the more reasonable for the case of the Sun; one concludes that in order to get the "right answer," one article takes a plausible spatial model but with the wrong opacity, while the other takes a plausible opacity within an unlikely model. Neither calculation works if all the ingredients are put in correctly! One might well wonder whether, without knowing the answer beforehand, an order-of-magnitude calculation of stellar structure can really be done. As "ball park" estimates, these calculations can be interesting and even valuable techniques for highlighting the underlying physics, but referring to them as order-of-magnitude analysis in order to convince the student that the numbers also come out right, without some idea as to how the orders of magnitude propagate, is surely misleading and certainly encourages loose thinking. In this sense, Nauenberg and Weisskopf's article sounds a warning bell, since their "fudge factor" enters into the final answer for the luminosity raised to the fourth power: Clearly, the result is sensitive to just how one does the calculation.

In short, there is currently a large gap between rough-and-ready "ball park" calculations (trivial, alluring but incomplete and giving rise to numerical results difficult to reproduce without a certain sleight of hand) and the full numerical model (complicated, inaccessible to the uninitiated, often giving results that are counterintuitive and difficult to explain in physical terms).

Yet the basic equations of stellar structure are simple.

II. SOLVING EQUATIONS WITH PHYSICAL UNDERSTANDING

Acton and Squire[12] recently discussed a very neat method for finding approximate solutions to scientific boundary value problems without going to the trouble of integrating (analytically or numerically) the equations involved.

Very briefly, they recognize that, in many problems of physical interest, it is possible to deduce from general principles how the solution should behave: for example, whether the function is convex or concave, rises or falls, is finite or zero at the boundaries, etc. Armed with this information, and knowing that few problems in the real world have solutions with unpleasant discontinuities or singular points, one then chooses the simplest one-parameter function that follows the general shape of the solution; in general, of course, this "trial function" (as the authors refer to it) will not satisfy the original equations over the entire range of the independent variable, but, unless one is unlucky, the parameter can be adjusted so as to satisfy them somewhere (if it cannot be, then presumably the chosen trial function is not a possible solution). Choosing this point to be some kind of "average point" (the authors call it the "collocation point"), one can then substitute the trial function into the original equations and find the value of the unknown parameter at the collocation point. One obtains thereby a "best fit" of the intuitive solution to the original problem. It is clear that, unless by a remakable stroke of luck (known sometimes as good intuition) the guess is the exact solution, different collocation points will give rise to different values of the parameter; however, if the guess is a reasonable approximation to the original problem, then the various fits will not be very different, and, conversely, the spread of the different solutions gives some idea of the reliability of the result.

In a certain sense, this method represents an engineering solution to differential equations—boundary regions are joined by a smooth curve, much in the way that boat builders allowed a flexible piece of wood to take up the optimal shape between two fixed points.

The procedure is best understood with reference to a simple problem possessing an analytical solution; the following example is not without astrophysical interest.

Consider a spherical mass of gas collapsing from rest under its own gravitation; we assume that when collapse begins the density is uniform with value ρ, and that the internal pressure can be neglected throughout the entire collapse.

The problem is to find the time taken to collapse to a point: This is the so-called free-fall time.

The equation of motion of an internal spherical shell of radius r is given by:

$$\frac{dv}{dt} = \frac{-GM}{r^2},$$

where v is its velocity, initially zero, and M is the total mass within r.

Substituting $dv/dt = (dv/dr)(dr/dt) = 0.5\, d(v^2)/dr$ into the equation of motion and integrating, one obtains

$$v = \frac{dr}{dt} = (2GM)^{1/2}\left(\frac{1}{r} - \frac{1}{r_0}\right)^{1/2},$$

where r_0 is the initial radius of the shell.

This equation *can* be integrated analytically to obtain the free-fall time exactly (see Rose,[3] for example); it turns out to be equal to

$$(3\pi/32G\rho)^{1/2} = 0.54/(G\rho)^{1/2}.$$

To find an approximate value for the free-fall time using the method of trial functions, we note that:

(1) The velocity must be zero at $t = 0$ and infinite at $r = 0$ (this curious circumstance comes about because the pressure has been neglected).

(2) The acceleration is finite at $t = 0$ and infinite at $r = 0$.

(3) Acceleration and velocity are monotonically rising functions.

A simple function that satisfies these boundary conditions is

$$r = r_0[1 - (t/\tau)^2]^{1/2},$$

where τ is the free-fall time. Note the square root and the square: Both are essential for the above boundary conditions to be satisfied. Of course, one can imagine more complicated functions.

Substituting this trial function into the relation for dr/dt, putting $\rho = 3M/4\pi r_0^3$, and rearranging, we emerge with a simple relation for τ:

$$\frac{1}{\tau} = \left(\frac{8\pi G\rho}{3}\right)^{1/2} \frac{\tau}{t}\left\{\left[1 - \left(\frac{t}{\tau}\right)^2\right]^{1/2} - \left[1 - \left(\frac{t}{\tau}\right)^2\right]\right\}^{1/2}.$$

A general property of the solution can be seen at once but seems surprising to some: For the case of uniform density, all shells arrive at the center simultaneously.

Choosing the collocation point to be at $t/\tau = 0.5$, the free-fall time evaluates to

$$\tau = 0.5/(G\rho)^{1/2},$$

which compares very favorably with the exact solution and is certainly easier to find by students unversed in the subtleties of transcendental integration.

Acton and Squire's book[12] shows that this degree of agreement is quite typical.

I show below how this elementary method can be applied to the problem of stellar structure for the particular case where the star has a uniform chemical composition. This discussion will throw some light on the question of why "ball park" analyses of stellar structure are bad, but not as bad as one has every right to expect.

III. REAL PHYSICS IN...

The equations of stellar structure (1)–(4) must be supplemented by information concerning the physical behavior of stellar material. I do not propose in this article to review this subject, since basic physics is covered more than adequately in many excellent texts; my purpose is to show how to apply this knowledge in a straightforward way to obtain approximate analytical solutions to the equations of stellar structure via the method of trial functions. Therefore, I shall reduce this part of the discussion to the strict minimum consistent with comprehension by nonspecialists; much of this material is simply taken from Schwarzschild.

Three extra relations put physical content into Eqs. (1)–(4).

(1) An equation of state. I shall take this to be the perfect gas equation,

$$P = nkT,$$

where n is the number of particles per unit volume and k is Boltzmann's constant.

It will turn out that, without sensible error and with the exception of the heavier elements whose abundance is, however, rather low, much of the interior of most stars can be taken to be virtually entirely ionized so that n must take account of both ions and electrons. This allows us to express the equation of state in terms of the chemical composition of the stellar material,

$$P = \rho kT(2X + 0.75Y + 0.5Z)/m_p, \tag{5}$$

where X is the hydrogen abundance expressed as a fraction of mass; Y is the helium abundance expressed in the same way; Z is the abundance of heavier elements expressed similarly; and m_p is the proton mass.

(2) The opacity χ of the stellar material. Three basic types of interacton of photons with ionized matter need be considered: electron scattering, scattering against ions without inducing transitions (free–free scattering), and scattering against ions in which bound electrons are raised

into the continuum (bound–free scattering; essentially only the heavier elements participate).

Electron scattering can be calculated exactly, and the opacity turns out to be independent of density and temperature:

$$\chi = 0.02(1+X)\ \mathrm{m^2/kg}\,. \qquad (6)$$

A very approximate analytical expression can be obtained for scattering against ions (a heuristic justification for the bound–free case may be found in Reddish[13]); it has the form

$$\chi = \chi_0 \rho T^{-3.5}\ \mathrm{m^2/kg}\,, \qquad (7)$$

with

$$\chi_0 = 10^{19}(1+X)(X+Y), \quad \text{for free–free scattering}\,,$$

$$\chi_0 = 7.0\times 10^{20}(1+X)Z\,, \quad \text{for bound–free scattering}\,.$$

In a given calculation, one should use whichever one of these gives the dominant contribution; this will depend on the internal structure, and so "ball park" analysis cannot be used to distinguish between the different cases. Note that numerical models usually rely on tabulated values of the opacity, or on analytical expressions of which the above are only a first approximation.

(3) Production of energy. Two basic processes only need be considered here: the transformation of hydrogen to helium via the proton–proton cycle or via the carbon–nitrogen cycle. The latter is more efficient at higher temperatures but is a function of the carbon and nitrogen abundance X_{cn}, which we shall take to be about $Z/3$. Nauenberg and Weisskopf[6] give a relatively simple *ab initio* derivation of the thermonuclear power as a function of temperature whose results one could use, but it is more convenient for the purpose of this article to use an approximate power law,

$$\epsilon = \epsilon_0 \rho X^2 (T/10^6)^{\nu}\,, \quad \text{for the pp cycle}\,, \qquad (8)$$

$$\epsilon = \epsilon_0 \rho X X_{cn} (T/10^6)^{\nu}\,, \quad \text{for the cn cycle}\,. \qquad (9)$$

The value of ϵ_0 can be tabulated for different temperature ranges: A very convenient table that I have used for the numerical estimates can be found in Ref. 1 (note, however, that his values are given in cgs units). Note that ν is in the range 3.5 to 6 for the pp cycle and in the range 13 to 20 for the cn cycle.

IV....REALISTIC RESULTS OUT

Even using the above analytical expressions (5)–(9), Eqs. (1)–(4) cannot be integrated analytically.

The boundary conditions, however, are quite straightforward. Even if one's intuition is not up to the task, the structure equations themselves indicate that mass and luminosity are rising functions, while pressure and temperature (as well as density) are decreasing functions. Moreover, the first derivatives vanish at the center of the star and tend to zero for sufficiently large values of r. One of the simplest functions whose derivative has the required properties is the Gaussian $\exp(-r^2/\lambda^2)$, where λ is a spatial decay constant to be determined for each of the variables; in the case of the mass, this decay constant is related to the stellar radius in a direct way. With the help of the Gaussian function, we may express the density, mass, temperature, and luminosity variations throughout the star using the following trial functions,

$$M = M_T[1-\exp(-r^2/\lambda^2)]\,, \qquad (10)$$

$$L = L_T[1-\exp(-r^2/\lambda_L^2)]\,, \qquad (11)$$

$$\rho = \rho_c \exp(-r^2/\lambda_\rho^2)\,, \qquad (12)$$

$$T = T_c \exp(-r^2/\lambda_T^2)\,, \qquad (13)$$

where M_T is the total mass; L_T is the total luminosity; ρ_c is the central density; and T_c is the central temperature.

Each of the variables has a different spatial decay constant, so that in this form the problem has seven unknown parameters instead of the usual four.

However, only four basic equations are available.

Collocation at several points is clearly indicated; this is contrary to the advice given by Acton and Squire, who consider it to be, in general, unduly complicated, but it turns out in this case to be very easy to apply. We can illustrate the procedure by using Eq. (2); substituting the trial functions 10 and 12 into Eq. (2), we obtain

$$\exp(-r^2/\lambda^2) \propto r\exp(-r^2/\lambda_\rho^2)\,. \qquad (14)$$

In the spirit of the trial function method, we assume that this relation will be satisfied at at least two points. The choice is not very critical, as we shall see below: Nevertheless, it would seem reasonable to choose points that span a substantial portion of the star—take $r/\lambda = 0.5$ and $r/\lambda = 2$, points that correspond, respectively, according to Eq. (10), to fractions 0.22 and 0.98 of the total mass. At these points, Eq. (14) becomes

$$\exp(-1/4) \propto \exp(-\lambda^2/4\lambda_\rho^2)/2\,,$$

$$\exp(-4) \propto \exp(-4\lambda^2/\lambda_\rho^2)\times 2\,.$$

This allows us to obtain λ_ρ in terms of λ,

$$\lambda^2/\lambda_\rho^2 = 1.37\,.$$

Using this result, we may obtain from Eqs. (1) and (3) in a similar way the decay constants for the temperature and for the luminosity distributions as functions of the decay constant for the mass distribution,

$$\lambda^2/\lambda_T^2 = 0.7\,,$$

$$\lambda^2/\lambda_L^2 = 2.34 + 0.7\nu\,.$$

Note that the decay constant for the luminosity is a decreasing function of the power index that defines the nuclear energy generation law; when the index is very high, as in the case of the cn reaction (see Ref. 1 for the detailed table), the energy producing volume shrinks to a small region around the center.

Using these decay constants, we may now rewrite Eqs. (1)–(4) in terms of the total mass, the total luminosity, the central density and temperature, and the decay constant for the mass distribution. Again, in the spirit of the method, we assume that the relations are satisfied at some value of r^2/λ^2, which we plausibly choose to be at the point where the mass attains half of its total value (the optimum choice for one-point collocation of exponentials, as recommended by Acton and Squire[12])—this corresponds to $r^2/\lambda^2 = 0.7$. In this way, after some straightforward manipulation, Eqs. (1)–(4) reduce to the simple algebraic identities (15)–(18), respectively:

$$\lambda = a(M_T/T_c)\,, \qquad (15)$$

$$1/\lambda^3 = b(\rho_c/M_T)\,, \qquad (16)$$

$$1/\lambda^3 = c(T_c \nu \rho_c^2/L_T)\,, \qquad (17)$$

$$\lambda = d(\rho_c L_T / T_c^4), \quad \text{for pure electron scattering,} \quad (18a)$$

$$\lambda = d(\rho_c^2 L_T / T_c^{7.5}), \quad \text{for ion scattering.} \quad (18b)$$

The quantities a, b, c, and d are functions of physical constants, and also depend on the way that the decay constants have been evaluated,

$$a = 0.2Gm_p/k(2X + 0.75Y + 0.5Z),$$

$$b = 1.6\pi,$$

$$c \approx [0.05\pi\epsilon_0/(2.34 + 0.7\nu)\times 10^{6\nu}]X^2,$$

for the pp cycle,

$$c \approx [0.05\pi\epsilon_0/(2.34 + 0.7\nu)\times 10^{6\nu}](XZ/3),$$

for the cn cycle,

$$d = 0.12\chi\{1 - \exp[-0.7(2.34 + 0.7\nu)]\}/ac,$$

for electron scattering,

$$d = 0.09\chi_0\{1 - \exp[-0.7(2.34 + 0.7\nu)]\}/ac,$$

for ion scattering.

It is now a trivial matter to derive rather simple analytical expressions for the mass decay constant (which is related to the radius of the star), for the total luminosity, as well as for the central temperature and density. Two different results are obtained, depending on whether electron or ion scattering is dominant.

It turns out that all the results may be written in the form

$$a^\alpha b^\beta c^\gamma d^\delta M_T^\mu$$

and so it is convenient to present them as a table giving the values of the powers α, β, γ, δ, and μ, for each of the scattering assignments (this is shown in Table I). Of course, the functional dependence on M_T is just a dimensional requirement; the interesting feature here is the ability to propagate through to the final result the dependence on physical constants, detailed mechanisms, etc.

Note that the functional dependence is very tightly linked to the assumption that energy is transferred through much of the star by radiation scattering only. If it is not, then one cannot rely on these expressions and one should work through the calculation using an appropriate equation in place of Eq. (4). This turns out to be particularly important for stars significantly less massive than the Sun, since a large fraction of the stellar envelope is then in convective equilibrium; one can easily show this using the expressions given in this article and a criterion given in Schwarzschild.[1] The difference in the dependence of, for example, the radius as a function of total mass, is spectacular. That any of these calculations work at all for the Sun (and this applies as much to the results of this article as to "ball park" estimates) is due to the lucky fact that only a small fraction of the solar envelope undergoes convection; stellar structure would be harder to teach if the Sun were less massive!

The object of the game is, ultimately, to reproduce the observable features of a known star. Therefore, let us postpone discussion of these expressions until later, and simply substitute the mass of the Sun, 2×10^{30} kg, to see how well they stand up to this simplest of tests—the observed luminosity is 3.8×10^{26} W, the radius is 7×10^8 m, and the surface temperature is 5770 K.

We should not impose *a priori* a particular nuclear energy generating law, which must emerge from the analysis itself; it is, therefore, advisable to tabulate first the results for several different values of the energy index ν (say, 6, 3.5, 20,and 13, the former two corresponding to the pp cycle and the latter two to the cn cycle). Similarly, the scattering mechanism should not be guessed in advance, and so for each value of ν the solar parameters have been calculated for the three principal mechanisms.

We can assume the chemical composition: $X = 0.73$, $Y = 0.25$, and $Z = 0.02$ are values used by Schwarzschild and so convenient to take in order to compare subsequently our results with his numerical calculations.

Finally, the analysis given in this article does not directly yield a stellar radius, but a spatial mass decay parameter λ. The radius can be said to be that distance from the center at which the mass has attained some large fraction of its total value, say, 0.999. With this criterion, the radius is just 2.6λ.

Table I. Parameters for the analytical expression of stellar quantities, written in the form $a^\alpha b^\beta c^\gamma d^\delta M_T^\mu$, for different scattering regimes.

	Scattering regime	α	β	γ	δ	μ
λ	Electron	$\frac{(\nu-4)}{(\nu+3)}$	$\frac{-3}{(\nu+3)}$	$\frac{1}{(\nu+3)}$	$\frac{1}{(\nu+3)}$	$\frac{(\nu-1)}{(\nu+3)}$
	Ion	$\frac{(\nu-7.5)}{(\nu+2.5)}$	$\frac{-4}{(\nu+2.5)}$	$\frac{1}{(\nu+2.5)}$	$\frac{1}{(\nu+2.5)}$	$\frac{(\nu-3.5)}{(\nu+2.5)}$
L_T	Electron	4	1	0	-1	3
	Ion	$\frac{(7\nu+22.5)}{(\nu+2.5)}$	$\frac{(2\nu+7)}{(\nu+2.5)}$	$\frac{-2}{(\nu+2.5)}$	$\frac{-(\nu+4.5)}{(\nu+2.5)}$	$\frac{(5\nu+15.5)}{(\nu+2.5)}$
ρ_c	Electron	$\frac{-3(\nu-4)}{(\nu+3)}$	$\frac{(6-\nu)}{(\nu+3)}$	$\frac{-3}{(\nu+3)}$	$\frac{-3}{(\nu+3)}$	$\frac{(6-2\nu)}{(\nu+3)}$
	Ion	$\frac{3(\nu-7.5)}{(\nu+2.5)}$	$\frac{(14.5-\nu)}{(\nu+2.5)}$	$\frac{3}{(\nu+2.5)}$	$\frac{3}{(\nu+2.5)}$	$\frac{(13-2\nu)}{(\nu+2.5)}$
T_c	Electron	$\frac{7}{(\nu+3)}$	$\frac{3}{(\nu+3)}$	$\frac{-1}{(\nu+3)}$	$\frac{-1}{(\nu+3)}$	$\frac{4}{(\nu+3)}$
	Ion	$\frac{10}{(\nu+2.5)}$	$\frac{4}{(\nu+2.5)}$	$\frac{-1}{(\nu+2.5)}$	$\frac{-1}{(\nu+2.5)}$	$\frac{6}{(\nu+2.5)}$

Table II shows the various possible "suns" corresponding to the different regimes assumed.

We notice immediately that whatever the imposed conditions, the temperature remains in the vicinity of 10^7 deg; for consistency, therefore, we can eliminate the higher values of ν in the case of the pp cycle and the lower value in the case of the cn cycle. We can distinguish between the two cycles by comparing their power output for the calculated central conditions—we find that in the case of the Sun the pp process dominates. Similarly, comparing the opacities at the calculated central conditions, one finds at once that the bound–free opacity is dominant.

These considerations allow us to identify which combination of parameters should be relevant; we note, in particular, that according to the table a more appropriate value for the nuclear energy index ν would be 4.5. We thus estimate a solar radius of about 4.5×10^8 m, radiating about 2.1×10^{26} W, with a surface temperature of about 5700 K (given by Stefan's law), to be compared with the actual values of 7×10^8 m, 3.8×10^{26} W, and 5770 K—the agreement is quite satisfactory. It it, in fact, better than this first comparison might indicate, since the calculation does not really pertain to the Sun as it is now, but as it was at the beginning of its existence—the analysis in this article has been carried out for a homogeneous star, and the present Sun is no longer in this happy state. To judge the quality of the appproximate analytical model, one should compare it to what is called in the astrophysical literature a "zero age" model: Just such a model is, in fact, presented in Schwarzschild (albeit using a more sophisticated opacity law, and with a suitable treatment of the outer stellar regions), and one finds that the luminosity of a homogeneous star having a solar mass and composition is about 0.6 of the present luminosity, i.e., about 2.3×10^{26} W. The radius of the zero-age model is marginally larger than the present one; note, however, that one should compare our "radius," which is that distance at which the mass is 0.999 of the final mass, with the equivalent quantity in the numerical model—this latter value is 0.88 of the solar radius, i.e., 6.2×10^8 km.

Finally, the central temperature and density given by Schwarzschild for the zero-age model are 1.2×10^7 and 7.7×10^4 kg/m^3; the corresponding values obtained by the trial function method (for $\nu=4.5$) are 1.1×10^7 and 7.9×10^4.

The method used in this article, simple as it is, yields solutions to the equations of stellar structure that are at the 30% level of the accuracy; this is often considered to be the preserve of numerical analysts.

V. THE FAULT, DEAR BRUTUS...

One of the striking features of the results evaluated for the Sun using the trial function technique is how insensitive certain of the final numbers are to the nuclear generation or opacity laws. Even the luminosity "only" spans a range of about 40. This is in some contrast to a statement sometimes made in defense of the "ball park" analyses when their results fail to match observed values, generally solar—the blame is put onto the opacity as the single item most likely to be wrong (the ball park calculations are usually done using the electron scattering opacity).

It is instructive to examine how the stellar parameters vary with the constants a, b, c, and d. For many values of ν (especially those appropriate to the pp reaction, dominant in solar type stars), many of the parameters are rather insensitive to the precise value of the constants (the luminosity is the single most important exception to this). The weakness of the trial function method lies in the apparently arbitrary way in which the collocation points are chosen. It is clear that using different collocation points on the Gaussian distributions will give rise to changes in the values of a, b, c, and d; however, it is easy to verify that these changes are considerably less than 50% and so have relatively little effect on the final parameters of the star; consequently, the effect of changes in, say, the opacity or energy generation laws show up in a reasonably clean manner.

It is amusing to apply the trial function technique to do a "proper" calculation using the model on which the "ball park" analyses are based, i.e., a linear change of all parameters from the center of the star to the outside. The parameters then take the form

$$M=M_T(1-r/R)\,, \tag{19}$$

$$L=L_T(1-r/R)\,, \tag{20}$$

Table II. Parameters for a solar mass object estimated using the analytical expressions of this article.

	Scattering regime	$\nu=6$	$\nu=3.5$	$\nu=20$	$\nu=13$
	Electron	3.7×10^8	3.4×10^8	3.1×10^8	3.7×10^8
Radius	Free–free	4.1×10^8	3.8×10^8	3.2×10^8	3.9×10^8
in m	Bound–free	4.8×10^8	4.8×10^8	3.4×10^8	4.3×10^9
	Electron	1.4×10^7	1.6×10^7	1.7×10^7	1.4×10^7
T_c	Free–free	1.2×10^7	1.3×10^7	1.6×10^7	1.3×10^7
	Bound–free	1.1×10^7	1.1×10^7	1.5×10^7	1.2×10^7
	Electron	1.4×10^5	1.9×10^5	2.5×10^5	1.5×10^5
ρ_c	Free–free	1.1×10^5	1.3×10^5	2.3×10^5	1.2×10^5
in kg/m^3	Bound–free	6.6×10^4	6.6×10^4	1.9×10^5	9.2×10^4
	Electron	2.0×10^{27}	2.1×10^{27}	2.0×10^{27}	2.0×10^{27}
L_T	Free–free	8.9×10^{26}	9.4×10^{26}	1.0×10^{27}	8.9×10^{26}
in W	Bound–free	2.0×10^{26}	2.1×10^{26}	2.4×10^{26}	2.1×10^{26}

$$\rho = \rho_c r/R , \tag{21}$$

$$T = T_c r/R , \tag{22}$$

with R being the stellar radius.

With this law, the boundary conditions are naturally not satisfied.

Using linear trial functions, the overall problem has only the usual four parameters; substituting Eqs. (19)–(22) in Eqs. (1)–(4) and collocating at $r/R = 0.5$ leads immediately to a set of equations whose form is identical to that of Eqs. (15)–(18b) (this is inevitable—dimensional similarity *oblige*), but with the constants a, b, c, and d now multiplied by numerical factors,

$$a_{\text{linear}} = a_{\text{Gaussian}} \times 10 ,$$

$$b_{\text{linear}} = b_{\text{Gaussian}} \times 0.3 ,$$

$$c_{\text{linear}} = c_{\text{Gaussian}} \times (2.3 + 0.7\nu)/2^{(2+\nu)} ,$$

$$d_{\text{linear}} = d_{\text{Gaussian}} \times 2^4\{1 - \exp[-0.7(2.3 + 0.7\nu)]\} .$$

These changes are hardly negligible, even when ν takes the values appropriate to the pp reaction, but one could live with them individually on an order-of-magnitude understanding.

Similarly, their effect on the value of the central temperature is not too catastrophic—it moves by somewhat less than an order of magnitude; this is why calculations such as that in Weisskopf's article,[5] which are based almost exclusively on the central temperature, are quite reliable.

However, folded into the expression for the luminosity, they change its value by rather more than two orders of magnitude. Here, we see, in pristine form the essential weakness of "ball park" analyses as applied to stellar structure: The orders of magnitude do not propagate through the calculation in a linear way, and so do not necessarily cancel at the end (a central assumption in order-of-magnitude estimates). One might be lucky, as in the case of the temperature, which turns out to depend on fractional powers of the dubious quantities, or very unlucky, as in the case of the luminosity, which depends on high powers of these same quantities. This is quite independent of the detailed values of physical constants such as opacity: Any attempt to improve the results of the calculations by adjusting the opacity constant, for example (on which the luminosity depends only in a roughly linear way), ignores that the luminosity is, in fact, much more sensitive to the way the model is built.

The ball park technique in whatever form gives poor numerical results, not just because the physical constants used are wrong, but because the model is a poor representation; while no one would deny that using the right constants is better than using the wrong ones, I submit that it is first essential to get the model right.

Nevertheless, it is some consolation to know that the error introduced by choosing a poor model is reasonably bounded so that the results obtained, while wrong, are not *ludicrously* wrong: This is why one obtains starlike parameters at the end of the calculation. However, one can obtain solarlike parameters only with the help of fudge factors (which it is best not to introduce in a systematic manner).

The analytical expressions for the stellar parameters as a function of the total mass can be used to illustrate a number of features of stellar structure—a couple of examples are sufficient to show the principle.

The well-known $L_T \propto M_T^3$ law for the case of electron scattering emerges almost immediately; however, this is hardly original.

Of greater interest is the equivalent luminosity mass relation for the case of ion scattering

$$L_T \propto M_T^{(5\nu + 15.5)/(\nu + 2.5)} ,$$

which, in the case of solarlike stars, yields $L_T \propto M_T^{5.46}$. This is close to the relation $L_T \propto M_T^{4.5}$ actually satisfied by stars less massive than about two solar masses (note in passing that the elementary discussion of the mass luminosity relation, based on the electron scattering law, is invalid in the mass range for which it is carried out—in the interest of simplicity, one tends to foist onto elementary students a calculation whose results are subsequently contradicted).

Electron scattering in stars having a normal chemical composition can only become important at high temperatures and low densities; since we have seen that even in the Sun ion scattering dominates, and since intuitively one would suppose that the density would rise with the mass, one wonders how any group of stars can enter the electron scattering regime and so satisfy the M^3 law.

If fact, this is a case where intuition is misleading. The opacity due to electron scattering is a constant, while the opacity due to ion scattering varies as $\rho/T^{3.5}$. Using the expressions for central temperature and density in the ion scattering regime, one finds immediately:

$$\text{opacity due to ion scattering} \propto M_T^{-(8+2\nu)/(\nu+2.5)} .$$

Consequently, the central opacity is a decreasing function of mass, and it is quite reasonable that at some value of the mass it should fall below the electron scattering opacity; the nature of the mass–luminosity law will then change.

The way in which the central density changes with mass depends on the scattering; the analytical expressions show that, while in the ion scattering regime the central density does rise with mass, when electron scattering becomes dominant, the central density can become an inverse function of the mass—this is a counterintuitive result.

Finally, the analytical expressions for the stellar parameters can be used for a variety of purposes, from finding the volume of a star undergoing nuclear burning, to obtaining upper and lower limits to the mass of a star involving fewer *ad hoc* assumptions about the critical temperature for nuclear burning (a fundamental defect of many "order-of-magnitude" estimates, as rightly pointed out by Greenstein[2]) or the pulsational instabilities to which gaseous masses are subject (it is very easy to find an expression for the maximum radiative flux that the surface of a star can support without losing its outer layers under the influence of radiation pressure, the so-called Eddington luminosity).

VI. CONCLUSION

The trial function method of finding approximate solutions to boundary value problems emerges as a viable way of analyzing the structure of homogeneous stars. The method requires no mathematical sophistication on the part of the student beyond the ability to differentiate a Gaussian function; it is nevertheless sufficiently flexible to allow the use of reasonably correct expressions for the rate of production of nuclear energy and for the opacity of the stellar material, and gives results that are within about 30% of numerical models. It is a useful quantitative adjunct to the type of qualitative discussion that is very popular in elementary textbooks. In this sense, the technique is to be preferred to the standard methods of introducing stellar structure via order-of-magnitude estimates, which are rarely within even two orders of magnitude of the right answer, and require particular care in the way orders of magnitude are neglected and the orders of magnitude of physical constants are chosen.

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Sunlight attenuation in nuclear winter

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A single algebraic expression is derived for the optical thickness of dust or smoke generated by a large-scale nuclear attack such as envisioned in nuclear winter studies. The formula is proposed as an addition to the set of "simple formulas" for the classroom study of nuclear effects. It may also be of use to analysts for first-order calculations.

I. INTRODUCTION

In a recent article in this Journal, Broyles[1] presents "some simple formulas and calculations to show students what elements are involved in estimating the amount of smoke that will be generated in a nuclear war and the resulting attentuation of sunlight." Broyles' formulas for the amount of smoke generated are indeed simple and reasonable. His formulas for sunlight attenuation, however, do not quite measure up to that standard. Diermendjian's formulas for Mie scattering are not "simple" as formulas (25)–(29) in Broyles' article testify. Further, since the resulting extinction coefficients are dependent on the smoke particle radius, the calculation for optical thickness can be carried out only for monosized particles rather than for a distribution of particle sizes as usually assumed in nuclear winter studies.

We offer here an alternate "simple" formula that overcomes both of the above objections and can be given a straightforward physical interpretation thus (hopefully) leading to a more intuitive understanding of the phenomenon of sunlight attenuation by smoke or dust.

II. AN EXPRESSION FOR OPTICAL THICKNESS

If we consider any sunlight interaction, whether absorption or scatter, to be a removal or extinction and if we ignore the added contribution from scattered sunlight, then the sunlight flux on the ground is related to the sunlight flux just above the sensible atmosphere by

$$\phi_{\text{ground}} = \phi_{\text{above atmosphere}}\, e^{-N_a\langle\mu\rangle}. \tag{1}$$

The product $N_a\langle\mu\rangle$ is the optical thickness OT, where N_a is the total number of particles in a column of air of unit area extending from the ground to the limit of the atmosphere and where $\langle\mu\rangle$ is the average extinction cross section or area presented for interaction. We note here that since we are ignoring scatter, the altitude location of the particles does not matter. (If scattering is treated, it matters a great deal.) The average extinction cross section $\langle\mu\rangle$ is found by integrating over the distribution of particle sizes,

$$\langle\mu\rangle = \int_0^\infty n(r)\mu(r)dr. \tag{2}$$

It is the cross-section variation with size that Broyles (and others) suggests may be modeled with Diermendjian's formulas, which are an approximation to Mie scattering theory. Ramaswamy and Kiehl[2] present the results of a Mie scattering calculation for 0.5-μm monochromatic light (near the peak of the visible spectrum). Their calculations yield the extinction efficiency Q_e, which is simply the ratio of extinction cross section to geometric cross section, that is,

$$\mu(r) = Q_e\pi r^2. \tag{3}$$

Pontier,[3] one of my graduate students, has also carried out these Mie calculations as well as the Diermendjian approximation described by Broyles. Pontier's Mie results are shown in Fig. 1.

Keeping in mind Broyles' purpose, "simple formulas and calculations to show students what elements are involved...," I suggest that, for classroom calculations, the extinction coefficient be taken as a constant, Q_e. The value of that constant must be defined by

$$\langle\mu\rangle = \int_0^\infty n(r)Q_e(r)\pi r^2\,dr = Q_e\int_0^\infty n(r)\pi r^2\,dr. \tag{4}$$