Another simple model for energy emission by black holes

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The qualitative results of Hawking and Page concerning the stability of black holes against energy emission are obtained by a direct application of Heisenberg's uncertainty principle.

In a recent paper, Kessler[1] presented a simple model for illustrating Hawking's proposal[2] that black holes can emit particles. He considered a black hole of mass M and Schwarzschild radius $R_c = 2GM/c^2$. The vacuum immediately surrounding the hole was taken to be the seat of virtual particle–antiparticle creation, the lifetime of each such pair being limited by the uncertainty principle; this led Kessler to estimate the spatial extent of each virtual pair. Since the process takes place in the gravitational field of the black hole, there is a gravitational gradient across a pair; a Roche limit type calculation for the stability of a particle–antiparticle pair led him to estimate the conditions under which the pair can be "disrupted"; one of the particles "falling" into the hole, the other being liberated.

While the paper was most stimulating to read, a number of quite arbitrary assumptions must be made in order to obtain the "right" answer, and I believe that Kessler's method is very likely to confuse the intelligent student. In this note, I propose an alternative approach which is at least as elementary and has fewer drawbacks. Before describing it, however, it may be useful to comment on a number of salient points in Kessler's paper. The reader is referred to it for details. A criticism has also recently been published by Peters.[3]

(1) The lifetime Δt and spatial extent Δs of a virtual pair: The lifetime is given by

$$\Delta t = \hbar/2\Delta E,$$

where ΔE is the total mass-energy of the virtual pair. In the paper, ΔE was taken as m_0c^2, which presupposes particles at rest since m_0 is the rest mass. However, the spatial extent was calculated from

$$\Delta s = \Delta t c.$$

This is most confusing. Clearly, particles at rest cannot leave, whatever the gravitational gradient; however, relativistically moving particles have $\Delta E \gg m_0c^2$.

(2) The binding force of a virtual pair was taken as due to Coulomb forces between a proton and an antiproton, and there are two unfortunate consequences. On the one hand, according to this argument, neutral particles (from, for example, neutron–antineutron pairs) should be able to leave with no trouble at all, whatever the mass of the black hole. On the other hand, the spatial separation turns out to be less than 1 fm, consequently the dominant forces are not Coulomb but nuclear. This could solve the neutral particle problem, since nuclear forces are charge independent; however, putting in the nuclear force reduces the critical mass considerably. One might argue that the method applies to electron–positron pairs. However, in that case, the critical mass turns out to be several orders of magnitude too high, since $M \propto 1/m_0$ in Kessler's formulas. In any case, the full calculations made by Page[4] are completely independent of interparticle interactions, as should, therefore, be heuristic approaches.

(3) The method gives no indication at all why low-mass particle emission is preferred to high-mass emission—this is quoted as a consequence of detailed analysis, which is unfortunate because the result is of fundamental importance.

(4) Critical mass for emission: In view of the previous remarks, the agreement with detailed calculations is due to a fortuitous choice of parameters. Moreover, I believe that one's satisfaction with the agreement is based on a misconception of the physical significance of the critical mass. In point of fact, one cannot really say that a 10^{15}-g black hole dissipates rapidly; the relevant quantity is the ratio of the lifetime of a black hole against "evaporation" to the Hubble time, and it turns out that only black holes of around 10^{15} g decay at a rate such that they are just now on the verge of giving up the last of their energy in a final burst. If the Hubble time were different, the critical mass would also be different, and this essential feature is nowhere to be seen in Kessler's model, which is independent of the Hubble time. This point is discussed in some detail by Peters.

(5) Finally, there is a wholly magical quality about the conclusions. For every particle which carries away energy, its antiparticle should apparently add an equal amount of energy to the black hole. However, the process is being invoked as a mean of reducing the black hole's mass. I submit that this must worry even the average student.

In point of fact, a "tidal stress" type of approach can be made somewhat more plausible, as has been done by Parker[5]; however, it involves an energy balance calculation rather than using the Roche limit, strictly applies only to essentially noninteracting particles such as photons or neutrinos, and explains the decrease of the black hole's mass through a particular (general relativistic) property of the immediate black hole environment which ensures that certain particle trajectories have negative energy relative to infinity, so that the incoming particle manages to decrease the mass of the black hole by a kind of "binding energy" effect. This latter point is by no means obvious.

The following method treats the "evaporation" of a black hole somewhat in the same spirit as the decay of radioactive nuclei: quantum-mechanical fluctuations permit energy to pass through a classically impenetrable barrier.

Consider a simple, isolated black hole. Its spatial extent is determined by the Schwarzschild radius $R_c = 2GM/c^2$, and in this sense one may say that the uncertainty in the measurement of the position of matter within it is given by $\approx 2R_c$. It therefore follows from the Heisenberg uncertainty

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principle that a momentum measurement is uncertain by the amount $\Delta p_x \Delta p_y \Delta p_z$, with

$$\Delta p_x \Delta p_y \Delta p_2 \Delta x \Delta y \Delta z \approx (\hbar/2)^3. \quad (1)$$

If we limit ourselves to relativistic particles (a qualitative justification follows later), we have

$$E = pc,$$

with

$$p^2 = p_x^2 + p_y^2 + p_z^2.$$

Therefore,

$$\Delta E = c\Delta p.$$

Now, the black hole is supposed at rest for simplicity. Consequently, Δp_x, Δp_y, and Δp_z represent not merely momentum fluctuations, but also the components of the fluctuating momentum Δp, so that to a small numerical factor we may write

$$\Delta p_x \Delta p_y \Delta p_z \approx \frac{4\pi}{3}(\Delta p)^3 = \frac{4\pi}{3}\left(\frac{\Delta E}{c}\right)^3.$$

Moreover, $\Delta x \Delta y \Delta z$ is the three-dimensional spatial uncertainty, and so is equal to the volume of the black hole:

$$\Delta x \Delta y \Delta z = \frac{4\pi}{3} R_c^3 = \frac{4\pi}{3}\left(\frac{2GM}{c^2}\right)^3.$$

Substituting in Eq. (1) and rearranging, one finds

$$\Delta E = \frac{\hbar}{GM}\left(\frac{9c^9}{1024\pi^2}\right)^{1/3}. \quad (2)$$

One can interpret the result in the following way: According to classical mechanics, energy which is within the black hole cannot leave it and go to infinity since the "escape velocity" within the Schwarzschild radius exceeds the speed of light. However, the uncertainty principle allows us to "borrow" a certain amount of energy ΔE for a time Δt. Providing that this time is sufficient for the energy to travel a distance equal to the Schwarzschild radius, a quantity of energy ΔE can "appear" just outside R_c. Once outside, and providing its velocity is c, it can then continue to infinity without further "help" from quantum mechanics.

This interpretation can also be used to derive Eq. (2) very easily. Δt must be of order R_c/c, so that

$$\Delta t \approx 2GM/c^2.$$

Since

$$\Delta E \Delta t \approx \hbar/2,$$

one finds

$$\Delta E \approx \hbar c^3/4GM,$$

which is numerically very close to Eq. (2).

This is simply the familiar tunnel effect, applied however to an unfamiliar situation. In the case of radioactive nuclei, there is a potential barrier of finite extent; in the black hole case, the gravitational potential barrier extends to infinity. However, as we have seen, energy does not have to tunnel to infinity (which would of course take an infinite time and so would allow only zero energy to do so), only to just outside R_c.

Energy fluctuations of value ΔE occur on the time scale $\Delta t = h/2\Delta E$, so that the "radiated" power P is given by

$$\begin{aligned} P &= E/\Delta t \\ &= 2E^2/\hbar \\ &= \frac{2\hbar c^6}{G^2M^2}\left(\frac{9}{1024\pi^2}\right)^{2/3} \\ &\approx 2 \times 10^{-2}\hbar c^6/G^2M^2. \end{aligned} \quad (3)$$

This result is to be compared to Page's numerical analysis:

$$P \approx 4 \times 10^{-4}\hbar c^6/G^2M^2.$$

Our result is an overestimate; one reason is that we have neglected the gravitational redshift, and another is given below. Nevertheless, certain features already emerge from Eqs. (2) and (3).

Types of particles emitted: For this mechanism to operate successfully, particles must appear outside R_c with velocity c. Now, ΔE represents a total energy fluctuation; for a particle of rest mass m_0 to materialize at all;

$$\Delta E \geq m_0 c^2$$

and for it to have velocity $\simeq c$:

$$\Delta E \gg m_0 c^2.$$

Consider first a black hole of mass 5×10^{16} g. Using Eq. (2), this gives

$$\Delta E \approx 0.5 \text{ MeV}.$$

This is below the threshold even for electron–positron production; consequently, black holes whose mass greatly exceed 5×10^{16} g can emit essentially no massive particles at all. The corresponding figure in Page's paper is $\approx 10^{17}$ g.

Similarly, we can see that for a black hole of mass 2×10^{14} g:

$$\Delta E \approx 120 \text{ MeV},$$

which is above the threshold for muon production; the corresponding figure in Page's paper is $\approx 4 \times 10^{14}$ g.

In this way, one can see how the mass of the black hole determines the types of particles emitted; for example: $M \ll 10^{14}$ g—massless particles, $e^{\pm}$, $\mu^{\pm}$, and heavier particles; $10^{14} \ll M \ll 10^{17}$ g—massless particles, $e^{\pm}$; $M \gg 10^{17}$ g—massless particles only.

The power radiated is a decreasing function of black hole mass. This can be understood qualitatively in the following way: The Schwarzschild radius of a black hole is proportional to its mass. The momentum, and therefore energy, fluctuation is inversely proportional to the spatial extent of the hole—more massive holes having a larger spatial extent, fluctuate less "violently" and so less energy can escape. Alternatively, one can say that the larger the black hole, the longer the duration of the quantum energy "debt" and so the smaller its allowed value.

This incidentally also indicates partly why the power estimate is too high: in point of fact, the energy should really tunnel to some distance well outside the Schwarzschild radius, since the escape velocity at the Schwarzschild radius is c.

Equation (3) can already be used to estimate the lifetime τ of a black hole against emission by this process:

$$\tau = Mc^2/P$$
$$= 50G^2M^3/\hbar c^4$$
$$\approx 3 \times 10^{-28}\, M^3 \text{ sec.}$$

It follows that the limiting mass of a primordial black hole which will now be just on the verge of giving up the last of its energy (which it will do quite violently, since $P \propto 1/M^2$) is given by

$$3 \times 10^{-28}\, M_{\lim}^3 = \tau_H,$$

where τ_H is the Hubble time. Putting 10^{18} sec for the Hubble time:

$$M_{\lim} \approx 1.5 \times 10^{15} \text{ g.}$$

This value is to be compared with Page's results, which fall around 5×10^{14} g—the precise value is a function of the particles emitted. Of course, the limiting mass varies only as $\tau^{-1/3}$ and so is relatively insensitive to discrepancies in the numerical constants.

It is interesting to see whether a "better" numerical agreement for the power radiated can be obtained in the same spirit, but by using some general relativity. Standard texts (see, for example, Bowler[6]) show that the geometrical cross section σ presented by a black hole to relativistic particles "fired" from infinity is greater than that given by the Schwarzschild radius:

$$\sigma = 27\pi G^2M^2/c^4.$$

This corresponds to an effective radius

$$R_\sigma = 27^{1/2}GM/c^2 \approx 5GM/c^2.$$

Particles which are in the zone $R_\sigma - R_c$ may still emit energy towards us. However, they are irretrievably lost and will enter the black hole. This is a result peculiar to general relativity and it cannot be obtained from any Newtonian argument. However, using it, the quantum-mechanical uncertainty in the position of a particle will be given by R_σ rather than by the Schwarzschild radius; in the quantum-mechanical sense, the spatial extent of the black hole would be operationally determined by firing test particles in all directions and thereby delimiting the region which absorbs everything.

Working through the calculations as before, and simply replacing R_c by R_σ, one obtains

$$\Delta E = \frac{\hbar}{GM}\left(\frac{9c^9}{16\,000\pi^2}\right)^{1/3},$$
$$p = \frac{2\hbar c^6}{G^2M^2}\left(\frac{9}{16\,000\pi^2}\right)^{2/3}$$
$$\approx 2 \times 10^{-3}\, \hbar c^6/G^2M^2,$$
$$M_{\lim} \approx 7 \times 10^{14} \text{ g.}$$

At first sight, it might seem extraordinary that such simple considerations can be used to obtain results which have required rather sophisticated quantum field calculations. On reflection, however, the agreement emerges as an example of what quantum mechanics is all about. Quantum field theory is based on the use of appropriate operators which obey certain rules; these rules are based on commutation relations, whose form is just a mathematical expression of the Heisenberg uncertainty principle. In some sense (but this is of course extremely unfair), quantum field theory may be looked upon as an indirect means of applying the uncertainty principle! A field theoretic statement of the phenomenon presented in this note is that particles are tunneling backwards in time, being scattered to forward time travel by a gravitational source which is beyond the event horizon. This is more rigorous (at least, for those who understand the jargon), but I feel that the basic physics does not emerge in as clear a way.

A final word of warning which should be given to all students. There has been a recent tendency to accept black holes as a going concern, and this is partly due to the fact that one has invented ways of making all sorts of sophisticated calculations about them (another reason is the quite understandable desire to associate exotic astronomical objects to exotic theories). However, the quantum mechanics which has been experimentally verified for the simple, stationary, and gravitationally flat world in which we live may or may not be applicable to the extreme conditions of the black hole environment—we simply do not know, and our ability to make formal calculations about a hypothetical object neither proves that the application is physically justified nor that the object exists. The subject is exceedingly stimulating and worth pursuing by those who have the taste; however, in the final analysis, only experiment or observation can decide.

There was considerable debate in the Middle Ages about the number of angels who could sit on the head of a pin. The discussions were closely reasoned and logically impeccable. However, the ability to argue the issue neither proved that angels had any desire to sit on the heads of pins, nor indeed that they existed.

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