The trajectory of an electron in a plasma

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Collisions in gaseous plasmas are fundamentally different from collisions in neutral gases because a charged particle interacts simultaneously with many others via the Coulomb potential. This difference is not as intuitive for students as the billiard ball like collisions of neutral particles. We present trajectories of electrons in a weakly coupled plasma obtained from particle simulations. The trajectories illustrate the concept of distant collisions and the variation of the free path with energy, which has major consequences in plasma physics. © *2008 American Association of Physics Teachers.*
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I. INTRODUCTION

A major difference between a plasma and a gas of neutral particles is the nature of collisions between particles. The collisions between neutral particles are similar to collisions between billiard balls, so that particle trajectories can be pictured as a succession of straight lines separated by abrupt changes in direction each time two particles approach each other by less than their diameter. The particle mean free path is the average distance traveled between such encounters.

The situation is completely different in a gas of charged particles. In this case each particle interacts simultaneously with a large number of distant particles via the Coulomb electric field, so that its trajectory experiences a superposition of many small perturbations. The mean free path is the average distance that the particle travels to accumulate enough small perturbations to produce a large variation in its motion.[1] The free path strongly increases with energy because fast particles are almost unperturbed by the Coulomb potential.

It has long been understood that long range Coulomb collisions play a major role in plasma physics. They make the velocity distribution function Maxwellian, equalize the temperatures of the different plasma components, and control the various transport coefficients, all this without the particles, on average, becoming close to each other. Quantifying these effects is crucial to understanding plasmas. The strong increase of the mean free path with energy is at the heart of some fundamental properties, such as the failure of classical calculations of transport coefficients in plasmas that are not strongly collisional.[2]

It is important that plasma physics students grasp the nature of the collisions in plasmas. Textbooks generally illustrate the collisions by hand-drawn sketches of the assumed trajectory of a particle. In this note we present several computed electron trajectories in a weakly coupled plasma obtained from simulations using N-body techniques usually applied to gravitational dynamics.[3] The computed trajectories serve to better illustrate the meaning of the mean free path and its variation with energy.

II. FUNDAMENTAL LENGTHS IN A PLASMA

Consider a completely ionized plasma consisting of n electrons of charge $-e$ and n ions of charge e per unit volume. The magnitude of the potential energy of two particles at distance r from each other due to the Coulomb interaction is $e^2/4\pi\epsilon_0 r$. When two particles come close enough that this energy becomes comparable to the kinetic energy of their relative motion, their trajectories are perturbed by a large angle. In a plasma at temperature T this angle is greater than $\pi/2$ if the colliding particles approach each other by a distance less than approximately $r_{\pi/2}$, which is given by

$$r_{\pi/2} = \frac{e^2}{12\pi\epsilon_0 k_B T}. \quad (1)$$

This distance is often called the strong interaction radius. More precisely, it is the impact parameter for which a proton deflects a typical electron (of kinetic energy $3k_BT/2$) by $\pi/2$, or equivalently the separation at which their interaction energy is twice the kinetic energy of their relative motion.

The mean interparticle distance $\langle r\rangle \simeq n^{-1/3}$ defines the typical energy of interaction $e^2/4\pi\epsilon_0\langle r\rangle$; dividing by k_BT yields the coupling parameter:

$$\Gamma = \frac{n^{1/3}e^2}{4\pi\epsilon_0 k_{\mathrm{B}} T}. \quad (2)$$

The plasma is said to be weakly coupled when $\Gamma \ll 1$. In this case the average interparticle distance is much greater than $r_{\pi/2}$. The condition $\Gamma \ll 1$ holds for most space plasmas, and we will make this assumption for the rest of the paper. In this case, the mean number of particles close enough to a given particle to undergo a large-angle deviation is about $nr_{\pi/2}^3 < \Gamma^3 \ll 1$. The probability that this configuration occurs is therefore very small.

The other fundamental length is the Debye length $L_{\mathrm{D}} = (\epsilon_0 k_B T/ne^2)^{1/2}$. It represents the distance beyond which the electrostatic field of a charge is screened by the other particles, which distribute themselves around a given charge to produce an oppositely charged sheath.[4] For the shielding to be effective and the plasma to be electrically quasi-neutral, two conditions must be met: its size must be larger than L_{D}, and there must be many particles in a volume of size L_{D}. Because $L_{\mathrm{D}} = \langle r\rangle/(4\pi\Gamma)^{1/2}$, the latter inequality is $4\pi\Gamma \ll 1$, which is similar to the weakly coupled inequality in Eq. (2), albeit somewhat stronger.

III. THE FREE PATH OF PLASMA PARTICLES

As discussed in Sec. II, a particle has, at any instant, a very small probability of suffering a close collision that produces a large angle deviation. At the same time it is subject to the Coulomb field of all the particles in a volume of size L_{D}, that is, about $nL_{\mathrm{D}}^3 = (4\pi\Gamma)^{-3/2} \gg 1$ particles, each of which

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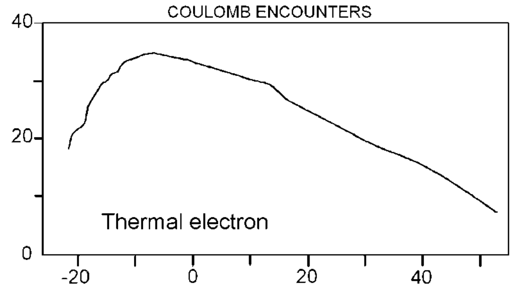

Fig. 1. Trajectory of a thermal electron in a plasma with $\Gamma = 0.02$. The figure is a projection of the three-dimensional trajectory, with axes labeled in units of the average inter-particle distance.

produce only a small perturbation because the mean inter-particle distance is much greater than $r_{\pi/2}$. We now ask what determines the mean free path: The rare and strong close collisions or the frequent and weak distant interactions?

First, let us consider the close collisions. Because any encounter closer than $r_{\pi/2}$ produces a large perturbation, the effective cross section for deviations due to such encounters is the surface of a disk of radius $r_{\pi/2}$, that is, $\sigma_{c,\mathrm{close}} = \pi r_{\pi/2}^2$. The corresponding free path is the length of a cylinder of section $\sigma_{c,\mathrm{close}}$ containing one particle, that is, $\ell_{fc} \sim 1/n\pi r_{\pi/2}^2$.

Let us now consider only distant collisions. Consider an electron passing by an ion (assumed at rest because of its large mass), with impact parameter $b > r_{\pi/2}$. Let $\Delta v_\perp$ be the variation in velocity perpendicular to the initial direction. The deflection produced by such a collision is equal to the Coulomb energy $e^2/(4\pi\epsilon_0 b)$ divided by the kinetic energy,[1] which for an electron of kinetic energy $3k_BT/2$ is

$$\frac{\Delta v_\perp}{v} \simeq \frac{2r_{\pi/2}}{b}, \tag{3}$$

if $b < L_D$ and is zero otherwise because the particles cannot feel the electric field at distances greater than the Debye length. These collisions induce deflections in random directions which tend to cancel, so that the average deflection is zero but the mean square is not. These collisions will eventually lead to a significant velocity change just as for a random walk. Because the number of collisions of impact parameter b producing a square deviation $(\Delta v_\perp/v)^2$ is proportional to $2\pi b$, the effective cross section for deviations due to distant interactions only is given by

$$\sigma_{c,\mathrm{distant}} = \int_{r_{\pi/2}}^{L_D} \left(\frac{\Delta v_\perp}{v}\right)^2 2\pi b db \simeq 8 \ln(\Lambda)\sigma_{c,\mathrm{close}} \tag{4}$$

with $\Lambda = L_D/r_{\pi/2} = 12\pi n L_D^3 \simeq \Gamma^{-3/2}$. We see that the effective cross section is the integral of the square deflections due to distant collisions at a given time multiplied by the area in which they take place. The corresponding mean free path is the distance the electron must travel to encounter enough particles so that the sum of $(\Delta v_\perp/v)^2$ equals unity; from Eq. (4) it is thus smaller than the mean free path for close collisions ℓ_{fc} by the factor $8 \ln(\Lambda)$.

The dimensionless parameter $\lambda = \ln(\Lambda)$, often called the Coulomb logarithm, is a measure of the importance of distant interactions with respect to close ones. In a weakly

Table I. Parameters used in the text for a weakly coupled plasma.

Γ	λ	$r_{\pi/2}/\langle r\rangle$	$\ell_D/\langle r\rangle$	$\ell_f/\langle r\rangle$
2×10^{-2}	5.9	6.7×10^{-3}	2.0	152

coupled plasma λ is much greater than unity. Thus the distant collisions are strongly dominant so that the mean free path is equal to:

$$\ell_f \simeq \frac{\ell_{fc}}{8\lambda} \simeq \frac{3\langle r\rangle}{4\pi\Gamma^2 \ln(1/\Gamma)}, \tag{5}$$

where we have approximated λ by $\ln(\Gamma^{-3/2})$. An exact calculation yields a value that is not very different.[1]

Note that we should distinguish collisions between electrons and ions from the collisions between like particles. The former only change the velocity direction but not its magnitude, whereas the latter change both the direction and amplitude of the velocity. This difference is due to the fact that for electron-ion collisions, one of the particles may be considered at rest (the massive ion), which is not the case for like particle collisions.[5]

IV. PLAYING WITH ELECTRONS

To illustrate these properties we show the trajectories of particles computed using a N-body simulation method. The system consists of $N = 2\times10^6$ particles ($N/2$ protons and $N/2$ electrons). The idea of the simulation is simple. We compute the Coulomb forces on each particle by the $N-1$ other ones. Each force is integrated over a small time step according to a leap frog scheme[6] which gives the updated velocity and position of each particle.[3] The initial velocity distribution function is a Maxwellian of temperature T for both electrons and protons. The particles evolve in a cubic volume whose faces reflect the particles back inside the domain to ensure a constant total number of particles. This method is expensive in computational time but allows us to

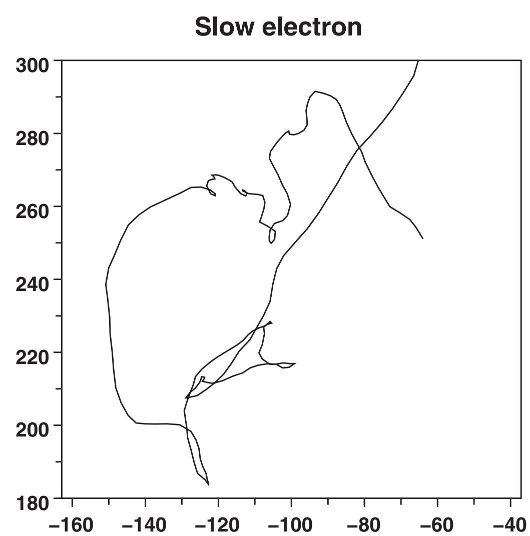

Fig. 2. Trajectory of a slow electron ($v \simeq 0.6v_T$) in a plasma with $\Gamma = 0.02$. The figure is a projection of the three-dimensional trajectory, with axes labeled in units of the average inter-particle distance.

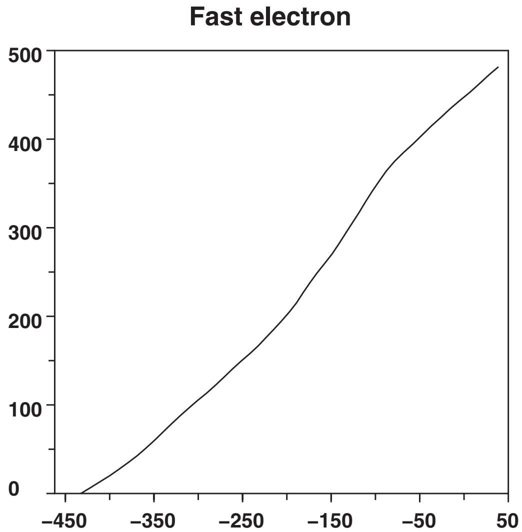

Fig. 3. Trajectory of a fast electron ($v \simeq 2.3 v_{\mathrm{T}}$) in a plasma with $\Gamma = 0.02$. The figure is a projection of the three-dimensional trajectory, with axes labeled in units of the average inter-particle distance.

include the slightest details of the physics into the simulation. In particular, the trajectory of all particles is known.

Figure 1, which is adapted from Ref. 7, shows the trajectory of an electron with the mean square speed. The figure is a projection in the plane (x, y) of a three-dimensional trajectory of an electron for a time during which it does not cross the plane boundaries of the volume. The plasma parameters are given in Table I.

A fundamental property of the mean free path is its dependence on energy. Equation (1) gives the strong interaction radius for a thermal electron having the mean square speed v_{T}. For an electron of speed v the strong interaction radius $r_{\pi/2}$ must be multiplied by $(v_{\mathrm{T}}/v)^2$, and the effective cross section by $(v_{\mathrm{T}}/v)^4$. Hence the mean free path is different from that of a thermal electron by the factor $(v/v_{\mathrm{T}})^4$. Hence an electron moving only three times slower or faster than average has a mean free path smaller or greater by two orders of magnitude. This difference is illustrated in Figs. 2 and 3, which show the trajectories of electrons with speeds equal to respectively 0.6 and 2.3 times the thermal speed.

We suggest showing these figures to students and asking them to calculate the various plasma scales from the value of Γ and to comment on the trajectories. Note that the mean free path given by Eq. (5) and in Table I is for a three-dimensional trajectory, but Figs. 1–3 are two-dimensional projections. Thus the mean free paths in the figures are less than the actual values by a factor of about 2. Figure 1 suggests a free path of $\approx 100\langle r\rangle$, in rough agreement with Eq. (5) which gives a value of $152\langle r\rangle$. Note that in Fig. 2 the speed $v \simeq 0.6 v_{\mathrm{T}}$ is approximate, because the particle undergoes many collisions, including electron-electron collisions that change its speed, so that its speed varies between about $0.3 v_{\mathrm{T}}$ and $0.9 v_{\mathrm{T}}$. This variation implies that its mean free path varies between about 1.23 and $100\langle r\rangle$. Students could also be asked to estimate the evolution of the electron velocity in time by looking at its trajectory. The aim is to spot the areas at which the particle is subject to more frequent collisions and to relate the collision frequency to the particle speed: Areas of higher collision frequency are areas of slower motion. The trajectory of the fast electron in Fig. 3 is simpler. Its mean free path is large ($\simeq 4.2 \times 10^3 \langle r\rangle$) and the electron undergoes few collisions so that its motion is almost straight and uniform.

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[1]Lyman Spitzer, Jr., *Physics of Fully Ionized Gases* (Wiley, Princeton, NJ, 1962), 2nd ed.

[2]See, for example, J. D. Scudder, Astrophys. J. **398**, 299–318 (1992); J. F. Luciani, Phys. Rev. Lett. **51**, 1664–1667 (1983). Because the mean free path of the fast particles can be several times the typical length scale of the plasma, their properties can be very different from the local particles. Consequently, a local classical calculation is likely to give incorrect results.

[3]More details of the *N*-body techniques for plasma simulations can be found in A. Beck, "N-body plasma simulations," Thesis Observatoire de Paris, 2008. The algorithm and its application to gravitational dynamics is discussed in W. Dehnen, "A hierarchical $\mathcal{O}$(N) force calculation algorithm," J. Comput. Phys. **179**, 27–42 (2002).

[4]Particles moving faster than the thermal ions can only be shielded by electrons; particles moving faster than the thermal electrons are not shielded at all and produce Langmuir waves. In the absence of equilibrium and/or for dimensions different from three, the shielding may be very different from the conventional one. See, for example, A. J. M. Garrett, "Screening of point charges by an ideal plasma in two and three dimensions," Phys. Rev. A **37**, 4354–4357 (1988), N. Meyer-Vernet, "Aspects of Debye shielding," Am. J. Phys. **61**, 249–257 (1993).

[5]For collisions between electrons the calculation must be done in the center-of-mass frame, and the mean free path for change in the velocity direction is greater by a factor of about 2.

[6]See T. Quinn, N. Katz, J. Stadel, and G. Lake, "Time stepping N-body simulations," eprint arXiv:astro-ph/9710043.

[7]Adapted from N. Meyer-Vernet, *Basics of the Solar Wind* (Cambridge U.P., Cambridge, 2007).