# 2

# Tool kit for space plasma physics

Most of the Universe is made of plasma. And yet, plasmas are very rare on the Earth, where solids, liquids and gases – the three primary states of matter – are ubiquitous (Fig. 2.1). These states are the result of a competition between thermal energy and intermolecular forces. In solids, the latter win, maintaining the atoms and/or molecules at nearly fixed positions, whereas thermal energy merely produces vibrations around these positions [9]. In gases on the contrary, thermal energy wins, making the particles almost completely free. Liquids are in between: the intermolecular forces are sufficiently strong to resist compression, but sufficiently weak to enable deformation and flow; it is not surprising that this intermediate state is less well understood than the other two [19].

Common experience and elementary physics tell us that we may transform a solid into a liquid by heating it; this weakens the bonds between molecules so that they may move slightly, enabling matter to change shape. This requires an amount of energy per molecule somewhat smaller than the binding energy. If the energy furnished exceeds the binding energy, the bonds break out completely, producing a gas of free atoms and/or molecules.

The *plasma* is the next state: the fourth, reached by furnishing enough energy to break the atoms themselves, or rather to kick off at least the outer atomic electron, producing a mixture of electrons and ions. For doing so, one has to heat or to compress, to bombard with energetic radiation or particles, or to subject the medium to high electric fields, as we shall see in more detail in Section 2.4. One (or several) of these ionisation agents acts in most regions of the Universe.

But (generally) such is not the case in the thin atmospheric layer of the small planet Earth, where human beings live. This medium is not ionised because it is a very special place: it is relatively cold; it is protected from the solar ionising radiation by an atmosphere; and when an atom happens nevertheless

Figure 2.1 Solids, liquids and gases abound on the Earth, but most of the Universe is made of plasma: the fourth state of matter. (Production of vapour; drawing by Jean Effel, *La Création du Monde*, 1971, copyright Adagp, Paris, 2007.)

to be ionised, the particle concentration is so high that ions and electrons meet frequently enough to recombine into neutral atoms. This is why our everyday experience of plasmas is so limited: we can see them occasionally in lightning, in some flames, inside fluorescent tubes or neon signs, but most of the visible plasmas lie farther away and are seen in sky displays, in auroras, comets and stars.

This chapter introduces briefly some tools of plasma physics that are essential for understanding the solar wind and its interaction with objects. In addition to introducing classical concepts, we give some hints on two subjects that lie at the frontier of traditional plasma physics: non-Maxwellian distributions, which are ubiquitous in the heliosphere – fooling our intuition and raising questions still unanswered – and ionisation processes. The aim is to furnish a tool kit for dealing with the major processes at work in the heliosphere, with the necessary limitations – in space and scope – of such a kit. We have privileged insight, at the expense of rigor and completeness. More may be found in several excellent texts, for example [17], [6], [10], [4], [18], [16] and [7].

## 2.1 What is a plasma?

In any gas there are always a few atoms or molecules that manage to lose one electron, producing some small degree of ionisation. Being ionised is therefore not sufficient to qualify as a plasma. A useful definition may be instead that a plasma is:

- a gas[1] containing charged particles (together with neutral ones), which
- is quasi-neutral
- and exhibits collective behaviour.

We will explain these three properties below. For simplicity, we consider a plasma made of electrons (charge $-e$, mass $m_e$) and one species of singly charged ions (charge $+e$, mass $m_i$) of equal concentrations $n$. We do not consider complex plasmas containing a large quantity of heavily charged ions or of dust particles, which are rare in the heliosphere. We also assume the particles to be non-relativistic and non-degenerate; relativistic and degenerate plasmas will be discussed briefly later.

The concentration $n$ is the average number of electrons (or ions) per unit volume. This assertion assumes implicitly that there are many particles in any volume considered, i.e. we shall consider spatial scales $L$ greater than the average distance between particles, whose order of magnitude is

$$\langle r \rangle \sim n^{-1/3} \quad \text{(average distance between particles).} \tag{2.1}$$

The temperature $T$ characterises the agitation of the particles. In thermal equilibrium, the particles' velocities along each space co-ordinate ($x$, $y$, $z$) are Gaussian distributed around zero (in the frame where the bulk of them is at rest), with mean square values $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = k_B T/m$ for a particle species of mass $m$; in this case, the average kinetic energy per particle is

$$m\langle v^2 \rangle/2 = 3k_B T/2. \tag{2.2}$$

However, an important property of space plasmas is their frequent lack of thermal equilibrium, even locally. Not only may electrons and ions have different bulk velocities and temperatures, but the particles' velocities may not be Gaussian distributed. In that case, one may still formally define a *kinetic temperature* for each particle species from (2.2), even though it is not a thermal equilibrium temperature. We shall return later to this point, which has basic applications in the solar corona (Section 4.6) and the solar wind (Section 5.5 and Problem 5.7.6). Meanwhile, we will assume that, even in the absence of thermal equilibrium, the particles have a typical random speed of the order of magnitude of $\sqrt{k_B T/m_{e,i}}$ (for the electrons and ions respectively).[2]

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[1]This restrictive definition is adequate for space plasmas. We do not consider plasma crystals [15].

[2]This assumption is not as trivial as it might seem. Consider for example a power law velocity distribution, so that the probability for the speed to lie in the range $[v, v + dv]$ varies as $v^{-\alpha}$ (with $\alpha > 0$) for $v_1 < v < v_2$, with $v_1 \ll v_2$. The most probable speed is $v_1$, whereas you can show as an exercise that the mean square speed – from which the kinetic temperature is defined – is of the order of magnitude of $v_1$ or $v_2$ depending on whether $\alpha$ is greater or smaller than 3, and the median speed is still very different. This example is extreme, since power law distributions are par excellence scale-free, but it is not academic since many processes produce similar distributions, as we shall see later in this book.

For each species of (non-relativistic and non-interacting) particles of number density $n$ and mass $m$, the pressure is determined by the average random kinetic energy as

$$P = nm\langle v^2\rangle/3 = 2w_{th}/3 \qquad \text{pressure} \quad (v \ll c) \tag{2.3}$$

where $w_{th}$ is the energy density of the particles; this is just the average flux of momentum along one space direction. This is equivalent to

$$P = nk_BT \tag{2.4}$$

with the kinetic temperature defined in (2.2). This definition of the pressure does not require the particles to be necessarily in thermal equilibrium. Note that in the simple plasma defined above (in which electrons and ions have the same number density $n$), the total particle pressure is the sum of the pressure of electrons and ions, that is $P = 2nk_BT$, where $T$ is their kinetic temperature (or the average of them if they are not equal).

### 2.1.1 Gaseous plasma

For an assembly of charged particles to qualify as a gas, the particles must move freely, which means that random motions should largely overrun mutual interactions. The latter involve the Coulomb force; for two particles of charge $\pm e$ distant by $r$, the energy of interaction is of modulus $e^2/4\pi\epsilon_0 r$. The plasma thus behaves as a gas if the energy of interaction of two particles distant by the average interparticle distance $\langle r\rangle$ is much smaller than the average kinetic energy per particle, i.e.

$$e^2/4\pi\epsilon_0\langle r\rangle \ll k_BT.$$

Substituting $\langle r\rangle \sim n^{-1/3}$, and introducing the coupling parameter $\Gamma$ defined as the ratio of the average energy of interaction to $k_BT$, we deduce the condition

$$\Gamma \equiv \frac{n^{1/3}e^2}{4\pi\epsilon_0 k_BT} \ll 1 \qquad \text{(gaseous plasma).} \tag{2.5}$$

In the solar wind, $\Gamma$ is of the order of magnitude of $10^{-8} - 10^{-7}$ at 1 AU, and varies weakly with heliocentric distance.

### 2.1.2 Quasi-neutrality

#### Debye shielding

Since charges of opposite signs attract each other, whereas charges of like signs repel each other, the Coulomb force tends to establish electric neutrality. The random agitation, however, mixes the particles, destroying this neutrality. The competition between both effects produces small regions that are non-neutral. The hotter the plasma, the greater the agitation and therefore the larger the maximum size of the non-neutral regions. On the other hand, the denser the medium, the greater the Coulomb force that keeps the plasma neutral, and therefore the smaller the size of the non-neutral regions.

To estimate this size, consider a region of size $L$ in which the electrons are strongly depleted, so that it contains a total electric charge of order of magnitude $Q \sim ne \times L^3$. This produces an electric potential at the boundary of the region, of order of magnitude

$$\phi \sim Q/(\epsilon_0 L) \sim neL^2/\epsilon_0.$$

For random agitation to produce spontaneously such a structure, the corresponding energy per particle $\sim k_B T$ must be at least equal to the potential energy per particle $e\phi$, i.e. $k_B T \geq ne^2 L^2/\epsilon_0$. We deduce (in order of magnitude) the maximum size of non-neutral regions

$$L_D = \left(\frac{\epsilon_0 k_B T}{ne^2}\right)^{1/2}, \tag{2.6}$$

the so-called *Debye length*.

Detailed calculations show that indeed when a charge is put in an equilibrium plasma, it attracts ambient charges of opposite signs and repels charges of like signs, so that it is surrounded by a region of size $L_D$ where the attracted particles are concentrated and the repelled ones are depleted, producing a charge distribution that shields the electrostatic field of the original charge. More precisely, the electrostatic potential at distance $r$ of a charge $q$ in an equilibrium plasma is

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r} e^{-r/L_{D*}} \tag{2.7}$$

where $L_{D*} = L_D/\sqrt{2}$ (because electrons and ions both contribute to the shielding). At distances $r \ll L_D$, the electric potential around the charge $q$ is nearly the Coulomb one, whereas at $r \gg L_D$, the charge is completely shielded by the charges of the ambient plasma, and the potential vanishes. Thus the plasma is quasi-neutral at scales greater than $L_D$.

This holds also for the charges of the plasma itself, and we have here a first hint as to a fundamental plasma property: its collective behaviour. Any charge in the plasma is 'dressed' by the other ones – a dressing of far-reaching consequences.

Numerically, $L_D \simeq 69\sqrt{T/n}$ in SI units, which comes to about 10 m in the solar wind at 1 AU from the Sun ($n \sim 5 \times 10^6$ m$^{-3}$, $T \sim 10^5$ K). Therefore, we have not to worry about the quasi-neutrality of the solar wind, except when dealing with scales smaller than tens of metres – a problem that occurs in the environment of space probes (Section 7.2).

It is worth noting that the Debye shielding requires several conditions to be met:

- a region of size $L_D$ must contain many particles, i.e. $nL_D^3 \gg 1$; with the definition (2.5) of $\Gamma$ and the expression (2.6) of $L_D$, this condition reads: $(4\pi\Gamma)^{3/2} \ll 1$;

- the electric disturbance produced by the charge $q$ on the ambient particles must not be greater than their average kinetic energy, otherwise they are not capable of shielding it;
- the charge $q$ is at rest;
- the plasma is in thermal equilibrium.

**Non-equilibrium plasma**

The latter condition is in practice rarely met in space plasmas. Consider first the case of partial equilibrium, namely when the different particle species are each in equilibrium but at different temperatures. Since the quasi-neutrality is ensured by the Coulomb force and destroyed by the random agitation, Debye shielding is mainly produced by the less agitated particles, so that $L_D$ is determined by the colder species.

In complete absence of equilibrium, the shielding is mainly provided by the slower particles of each species. More precisely (Problem 2.5.1), if the charge produces a sufficiently small perturbation, then the shielding length $L_{D*}$ is determined by the average of $1/v^2$ for each species, as

$$1/L_{D*}^2 = 1/L_{D_e}^2 + 1/L_{D_i}^2 \tag{2.8}$$

with

$$1/L_{D_{e,i}}^2 = ne^2 \langle v^{-2} \rangle_{e,i} / (\epsilon_0 m_{e,i}) \tag{2.9}$$

where the subscripts $e$ and $i$ stand for electrons and ions respectively. At equilibrium at temperature $T$, $\langle v^{-2} \rangle = m/(k_B T)$, so that the shielding length reduces to $L_{D*} = L_D/\sqrt{2}$ with $L_D$ given in (2.6).[3]

In essence, Debye shielding is not determined by the random kinetic energy of the particles, but by the average of the inverse of that kinetic energy.

**Non-linear shielding**

What happens when, in addition to the plasma not being in equilibrium, the electric disturbance produced by the charge is large, i.e. the Coulomb potential energy is not small compared to the kinetic energy? In that case, we shall see later that the particles produce a different contribution to the shielding, depending on whether they are attracted or repelled.[4] The resulting distribution of the attracted particles then depends on the geometry of the problem (see for

[3]For each species of mass $m$ and temperature $T, \langle v^{-2} \rangle = \int_0^\infty dv e^{-mv^2/2k_BT} / \int_0^\infty dv v^2 e^{-mv^2/2k_BT}$.

[4]This is so because in order to shield the charge $q$, repelled particles have just to decrease their number density, which can be achieved by a mere deviation of their trajectories; attracted particles, on the other hand, have to increase their number density in order to shield the charge $q$, which requires some of them to change their incoming trajectories into closed orbits around $q$ – a performance that requires collisions and cannot be achieved in the absence of equilibrium.

example [11] and [8]). We shall return to this point in Section 2.3, and shall see examples of application in Section 7.2, when calculating the electric charge of objects immersed in the solar wind.

### Shielding of a moving charge

What happens if the charge $q$ is moving? The answer depends on the value of its speed $v$ compared to the most probable speeds of the plasma particles, whose order of magnitude is $(k_BT/m_{e,i})^{1/2}$ for respectively electrons and ions. Because electrons are much lighter than ions, these speeds satisfy the inequality $v_{thi} \ll v_{the}$. If the speed $v \ll v_{thi}$, then the plasma electrons and ions are fast enough to keep up with the charge motion, so that the shielding is not affected. On the other hand, if $v_{thi} \ll v \ll v_{the}$, then the plasma electrons are still fast enough to keep up with the charge motion, but the ions move too slowly to do so. In that case, the shielding is provided by the electrons only. Finally, if the charge $q$ moves faster than the electrons (and the ions), then the bulk of the plasma particles cannot catch up with it, and therefore cannot shield it. Instead, the charge motion produces plasma waves, a novel kind of dressing to which we shall return in Section 2.3.

### Timescale for shielding

This disappearance of Debye shielding (or rather its transformation into a new kind of dressing) occurs when the charge moves fast from the point of view of the electrons. A related problem is what happens when a charge $q$ is suddenly put in a plasma initially at equilibrium. The plasma particles will take some time to distribute themselves in order to provide shielding. How long? Electrons, moving faster than ions, are the first to shield the charge. For doing so they must travel a distance of the order of $L_D$. At the most probable speed $v_{the}$, this takes the time $\tau \sim L_D/v_{the}$. With the expression (2.6) of $L_D$ and $v_{the} \sim (k_BT/m_e)^{1/2}$, we find $\tau \sim \left(\epsilon_0 m_e/ne^2\right)^{1/2} \equiv 1/\omega_p$, where $\omega_p$ is the so-called (angular) *plasma frequency*, a basic plasma parameter to which we shall return later.

We get here a second hint as to the collective behaviour of plasmas. Not only are the charges dressed, but this dressing is highly dynamic, with a timescale of the order of magnitude of $1/\omega_p$.

This has an important implication. Consider an electromagnetic wave incident on a plasma. The variable electric field of the wave tends to destroy the plasma quasi-neutrality. But if the wave frequency is smaller than the plasma frequency, the disturbance has a timescale large enough that the plasma particles are capable of catching up with it and of restoring the quasi-neutrality. If they succeed, the electric field is cancelled and the wave does not propagate in the plasma. We shall see in Section 2.3 that, indeed, electromagnetic waves propagate in a plasma only at frequencies greater than the plasma frequency.

### 2.1.3 Collisions of charged particles

We now come to a further plasma property, which concerns collisions between particles.

Collisions serve to achieve equilibrium. They determine not only the time required to restore thermal equilibrium after a perturbation, but also the transport coefficients which control the response of the medium to various gradients in macroscopic properties:

- the diffusion coefficient, which determines the transport of particles in response to a gradient of concentration;
- the viscosity, which determines the transport of momentum in response to a gradient of velocity;
- the thermal conductivity, which determines the transport of heat in response to a gradient of temperature;
- the electric conductivity, which determines the transport of electric charge in response to an electric field.

The collisions thus play an important role, and a major difference between plasmas and neutral gases is the cross-section for particles' collisions. This has profound implications for plasma behaviour, which contradict the intuition acquired with neutral gases.

#### A reminder on collisions in neutral gases

Collisions between neutral particles have much in common with those of billiard balls. Macroscopic neutral particles collide when they come into contact, namely when they come closer than about their physical size. More precisely, two spheres of radius $r$ collide when their centres come closer than $2r$, so that their cross-section for collision is the area of a circle of radius $2r$, i.e. $4\pi r^2$. The 'size' of an atom or a molecule relevant for collisions is not so clear-cut as the one of a billiard ball since the interaction involves induced dipoles in the distribution of electrons, a distribution determined by quantum mechanics. So, the cross-section for collisions between neutral atoms or molecules is somewhat greater than the 'billiard ball' value (taking for $r$ a typical atomic size – see Section 2.4.1), but not by more than one order of magnitude. This yields the crude estimate

$$\sigma_{col} \sim 10^{-19} \text{ m}^2. \tag{2.10}$$

As in the case of billiard balls, most collisions between neutral atoms and molecules result in a large variation in momentum and energy (Fig. 2.2, left).

The mean collisional free path of particles is the average distance they have to travel in order to undergo one collision. A particle of cross-section $\sigma_{col}$ travelling a distance $l$ encounters all the particles contained in a cylinder of section $\sigma_{col}$

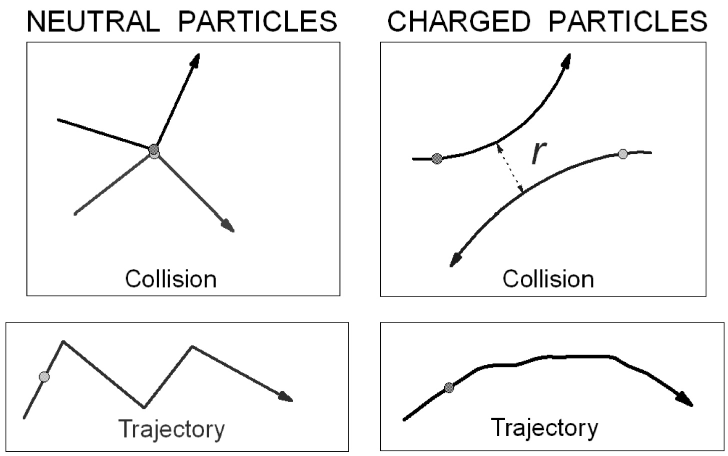


Figure 2.2 Collisions between neutral particles (left panels) and between charged particles (right panels). Collisions between neutrals occur, crudely, when they come into contact, and generally produce a large change in trajectory (top left panel). In contrast, charged particles interact even at large (but smaller than $L_D$) distances, via the Coulomb force (top right panel). The corresponding trajectories are sketched in the bottom panels.

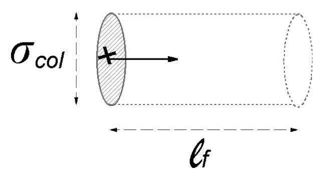


Figure 2.3 The collisional free path is the average distance travelled by a particle to undergo one collision. A particle of cross-section $\sigma_{col}$ for collisions with particles of concentration $n$ has the free path $l_f = 1/(n\sigma_{col})$.

and length $l$ (Fig. 2.3), i.e. $n \times \sigma_{col} \times l$ particles of number density $n$. The collisional free path $l_f$ corresponds to one collision, i.e.

$$l_f = (n\sigma_{col})^{-1} \qquad \text{collisional free path.} \tag{2.11}$$

Near the surface of the Earth, the typical distance between particles is about $3\times10^{-9}$ m, so that with the cross-section (2.10), the mean free path for collisions is of the order of magnitude 1 μm.

The *collision frequency* is the inverse of the average time between two collisions, that is the time for travelling the distance $l_f$. With a relative speed $v$, this yields

$$\nu_{col} = v/l_f = nv\sigma_{col}. \tag{2.12}$$

### Collisions between charged particles and neutrals

The cross-section for collisions between electrons and neutrals is given in order of magnitude by the above value (2.10). Because of their small mass, electrons move much faster than neutrals, so that the relative velocity is about their most probable speed $v_{the} \simeq (2k_BT/m_e)^{1/2}$. Hence the frequency of collision of electrons with neutrals of concentration $n_n$ is given by substituting $v \sim v_{the}$ and the cross-section (2.10) in (2.12), which yields

$$\nu_{en} \sim 5 \times 10^{-16}\, n_n \sqrt{T}, \qquad (2.13)$$

a result we shall use in the context of planetary ionospheres (Section 7.1).

For collisions of ions with neutrals, the induced dipoles play a more important role, and the cross-section depends somewhat on the particle energy. At a small enough temperature (as for example near comets), the cross-section is proportional to the inverse of the relative speed, and a useful approximation for the frequency of collisions of ions with neutrals of concentration $n_n$ is

$$\nu_{in} \sim 3 \times 10^{-15}\, n_n \sqrt{m_p/m_i}. \qquad (2.14)$$

### Coulomb collisions

The mutual interaction of charged particles is basically very different. Consider two charges approaching each other (Fig. 2.2, right-hand panel). Since they interact via the Coulomb force, each ‘encounter’ generally deviates their trajectories, provided the particles come closer than the Debye length. (Farther away, the charges are shielded by the ambient plasma and no longer interact.) Each such encounter may thus be considered as a ‘collision’.

What is the distance of closest approach required to produce a large perturbation in trajectory? Whatever the relative sign of the charges (in Fig. 2.2 the two charges are of like sign), the perturbation is large if the potential energy of interaction is at least equal to the average kinetic energy, i.e. $e^2/4\pi\epsilon_0 r \geq k_BT$, hence if the distance of closest approach is smaller than

$$r_L \equiv \frac{e^2}{4\pi\epsilon_0 k_B T} \qquad \text{(Landau radius)} \qquad (2.15)$$

in order of magnitude. Any encounter closer than this distance will result in a large perturbation in trajectory. We deduce that the effective cross-section for collisions producing a large perturbation is $\sigma_C \sim \pi r_L^2$, and the corresponding free path is

$$l_f \sim \left(n\pi r_L^2\right)^{-1} \qquad \text{(mean free path for large perturbations).} \qquad (2.16)$$

What happens if the plasma is not in equilibrium? We may apply the same reasoning, but now $k_BT$ has to be replaced by the kinetic energy of the particle,

$mv^2/2$, for a particle of (relative) speed $v$ and (reduced) mass $m$. The effective distance for collisions producing a large perturbation is therefore in that case

$$r_{ef} = \frac{e^2}{4\pi\epsilon_0 \times mv^2/2} \propto v^{-2}.$$

An important result emerges: the faster the particle, the smaller the cross-section for collisions $\sim \pi r_{ef}^2 \propto v^{-4}$. Hence, fast particles undergo very few collisions. We shall return later to this point, which has basic consequences for plasma behaviour. Another interesting result is that if electrons and ions have similar temperatures, their collision cross-sections are similar, so that they have similar mean free paths. Since the collision frequency varies as $v/l_f$ and electrons (being much lighter) move much faster, they have a much greater collision frequency.

How frequent are these close encounters producing large perturbations? Most particle encounters occur at distances of closest approach of the order of magnitude of the average distance between particles $\langle r \rangle \sim n^{-1/3}$. From the expression (2.15) of $r_L$ and the definition (2.5) of the coupling parameter $\Gamma$, we have

$$r_L/\langle r \rangle = \Gamma. \tag{2.17}$$

Since $\Gamma \ll 1$, the distance $r_L$ for producing a large perturbation is much smaller than the average interparticle distance, so that close encounters are very rare. Most encounters occur at much larger distances, resulting in small perturbations, so that the trajectory of charged particles is made of a succession of small deviations, rather than the zigzag path of neutrals (Fig. 2.2, bottom).

Figure 2.4 illustrates this property in a more realistic way. It shows the trajectory (projected on a plane) of a typical electron in a plasma with $\Gamma = 0.02$, from a numerical simulation [1] handling $2 \times 10^6$ particles in a box of size $10^2$ times larger than the average distance between electrons.

### Mean free path for collisions of charged particles

As a result of the numerous encounters at large distances, the cross-section for collisions of charged particles is greater than the value $\pi r_L^2$, which takes into account only the rare close encounters producing a large perturbation.

Consider the simple case of an electron that passes near a positive ion, with impact parameter $p$ and velocity $\mathbf{v}_e$ and undergoes a small deviation (Fig. 2.5). Because of the large ion mass, we suppose it to be at rest. Most of the deviation of the electron takes place in the part of its trajectory where it is closest to the ion, i.e. at a distance of order of magnitude $p$ from the ion, namely as it travels a distance of about $p$ on each side of the ion, i.e. the distance $2p$ parallel to $\mathbf{v}_e$; this takes the time $\delta t = 2p/v_e$. In this part of the path, the Coulomb force on the electron is $F_\perp \simeq e^2/4\pi\epsilon_0 p^2$, roughly perpendicular to the original electron velocity $\mathbf{v}_e$. During the time $\delta t$, this force produces a change $\delta v_\perp$ in the

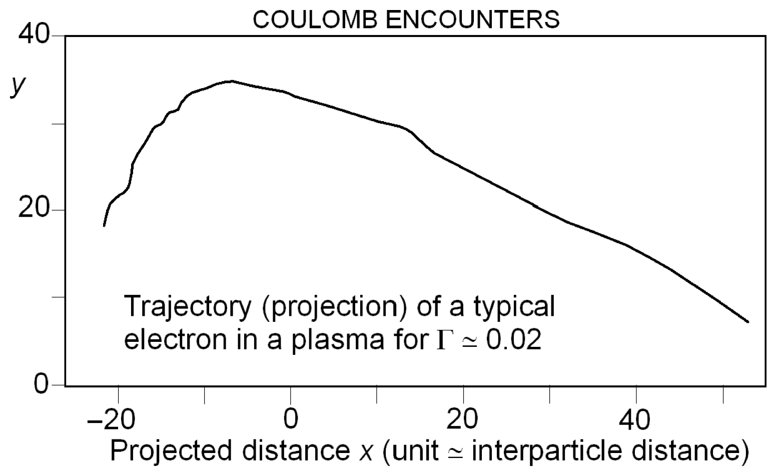


Figure 2.4 Typical trajectory (projected on a plane) of an electron in a plasma with $\Gamma = 0.02$, from a numerical simulation [1]: a box of size 110 arbitrary units contains $2 \times 10^6$ Maxwellian particles (electrons and ions); the trajectory shown is that of an electron having roughly the most probable speed. The mean free path (2.22) is nearly equal to the size of the box. (Courtesy A. Beck.)

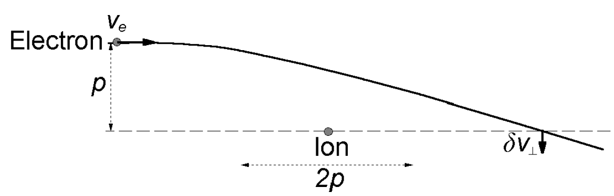


Figure 2.5 An electron of velocity $\mathbf{v}_e$ passing at distance $p$ from an ion (of negligible velocity) and undergoing a small deviation.

electron velocity (perpendicular to $\mathbf{v}_e$) given by $m_e \delta v_\perp \simeq F_\perp \delta t$. Rearranging, this yields[5]

$$\delta v_\perp = v_e \times r_{Le}/p \quad \text{with} \quad r_{Le} = \frac{e^2}{4\pi\epsilon_0 \times m_e v_e^2/2}. \tag{2.18}$$

Statistically, the deviation may be in either sense with equal probability; hence the individual deviations do not add, but their squares do, as in a random walk. We thus calculate the mean total variation $\langle \Delta v_\perp^2 \rangle$ during a given time $\Delta t$, by integrating over encounters of various impact parameters $p$ occurring during this time. From (2.18), each encounter of impact parameter $p$ produces $\delta v_\perp^2 = (v_e r_{Le}/p)^2$. The number of encounters of impact parameter in the range

[5] An exact calculation turns out to yield the same result.

$[p, p+dp]$ (crossing the area $2\pi p dp$) during the time $\Delta t$ is $dN = nv_e \times 2\pi p dp \times \Delta t$, so that

$$\langle \Delta v_\perp^2 \rangle = \int \delta v_\perp^2 \times dN = 2\pi \, n r_{Le}^2 \, v_e^3 \Delta t \int dp/p. \tag{2.19}$$

For impact parameters $p < r_{Le}$, the deviation is large, contrary to our assumption, whereas for $p > L_D$ the charges do not interact because of Debye shielding. Hence the integral (2.19) must be calculated in the range $r_{Le} < p < L_D$, which yields the factor $\ln(L_D/r_{Le})$.

The collision frequency is the inverse of the time $\Delta t$ needed to produce a large deviation, i.e. to produce $\langle \Delta v_\perp^2 \rangle \simeq v_e^2$. Substituting this value into (2.19) yields the collision frequency $1/\Delta t$ between electrons and (singly charged) ions

$$\nu_{ei} \simeq nv_e \times 2\pi r_{Le}^2 \ln(L_D/r_{Le}) \tag{2.20}$$

whence the collisional free path

$$l_f \simeq \left[ n \times 2\pi r_{Le}^2 \ln(L_D/r_{Le}) \right]^{-1}. \tag{2.21}$$

The mean value at equilibrium may be estimated by replacing $m_e v_e^2/2$ by the average kinetic energy $3k_BT/2$. From (2.15) and (2.18), we have $r_{Le} \simeq 2r_L/3$ and $L_D/r_{Le} \simeq 3/\left(4\sqrt{\pi}\Gamma^{3/2}\right)$, so that (2.21) yields the mean electron free path for collisions

$$l_f \simeq \left[ n \times (4\pi/3) \, r_L^2 \ln(1/\Gamma) \right]^{-1} \tag{2.22}$$

where $r_L$ is given by (2.15) and $\Gamma$ by (2.5).[6] One can verify (Problem 2.5.2) in Fig. 2.4 that for a typical electron the velocity direction indeed changes significantly when the particle has travelled a distance given roughly by (2.22). This equation yields approximately (in SI units)

$$l_f \simeq \frac{10^9}{\ln(1/\Gamma)} \times \frac{T^2}{n}. \tag{2.23}$$

Comparing (2.22) with (2.16), we see that the cumulative effect of the numerous small deviations decreases the free path (and increases the collision frequency) by a factor of order of magnitude $\ln(1/\Gamma)$. For typical space plasmas that we shall encounter in this book, this factor lies approximately between 10 and 20.

In the solar wind at 1 AU from the Sun, we have $n \sim 5 \times 10^6$ m$^{-3}$ and $T \sim 10^5$ K, so that the typical distance between particles is about 5 mm, whereas the mean free path is about 1 AU; collisions are thus very rare in the solar wind.

We considered for simplicity an electron encountering a singly charged ion. For an electron encountering an ion of charge $Ze$, the Coulomb force is greater by the factor $Z$, and therefore so is the radius $r_L$, producing an electron-free path smaller by the factor $1/Z^2$.

[6] We have approximated $\ln(0.6/\Gamma)$ by $\ln(1/\Gamma)$, which yields a very small error since $\Gamma \ll 1$.

### Timescales for equilibrium

The scales $1/\nu_{ei}$ and $l_f$ represent respectively the average time and distance for an average electron to change significantly the direction of its velocity due to the collisions with ions. Because of the large difference in mass between electrons and ions, this barely changes the particle energy.

Consider now collisions between electrons themselves. The calculation is slightly different since one can no longer assume one particle to be at rest, but we can make the calculation in the frame of the particles' centre of mass (using the reduced mass $m_e/2$), and the collision frequency is of the same order of magnitude. The major difference is that for particles of like mass, the collision now changes also the particle energy. Hence the values of $\nu_{ei}$ and $l_f$ calculated above represent respectively (in order of magnitude) the collision frequency and the free path of electrons for change in *speed direction* (because of encounters with *ions and electrons*) and in *energy* (because of encounters with *electrons*).

Consider now the collisions between two ions. The result is the same as for collisions between two electrons, just replacing the electron properties by those of ions. Hence, if the temperatures are similar, the mutual collision frequency of ions is smaller than the above value by a factor equal to the ratio of their most probable speeds, that is about $(m_e/m_i)^{1/2}$, whereas the free path is the same as above.

### Photon mean free path versus particle mean free path

It is interesting to compare the effective cross-section of electrons for collisions with charged particles $\sigma_C$, which is about one order of magnitude greater than $\pi r_L^2$ (because of the numerous large-distance encounters), with the effective cross-section of electrons for interaction with photons (the Thomson cross-section), given in (1.11). From (1.12) and (2.15), the ratio of both cross-sections is

$$\frac{\sigma_C}{\sigma_T} > \left(\frac{r_L}{r_e}\right)^2 \simeq \left(\frac{m_e c^2}{k_B T}\right)^2 \tag{2.24}$$

where we have substituted the so-called classical electron radius $r_e = e^2/\left(4\pi\epsilon_0 m_e c^2\right)$. This ratio is much greater than unity for non-relativistic plasmas. Hence, plasmas interact more with plasmas than with radiation, and photon mean free paths in plasmas are generally much greater than charged particle free paths.

## 2.1.4 Plasma oscillations

Consider a volume of plasma that is initially quasi-neutral, and imagine that you displace all the electrons along the $\mathbf{x}$ axis by a distance $x$ (Fig. 2.6). This produces a charge per unit volume equal to $\pm ne$ in two slabs of width $x$ at the extremities; the electric field is equal to that between two capacitor plates of

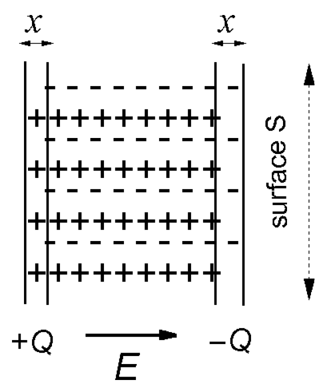


Figure 2.6 When electrons (in a plasma of density $n$) are displaced by $x$, the charge separation produces the electric field of a capacitor whose plates carry the charge of the electrons (or ions) contained in a plasma slab of width $x$, i.e. $\pm nex$ per unit area.

charge per unit area $\pm nex$, i.e

$$E = nex/\epsilon_0.$$

Each displaced electron is subject to a force $-eE$ along $\mathbf{x}$, and moves according to

$$m_e \partial^2 x/\partial t^2 = -eE \quad \Rightarrow \quad \partial^2 x/\partial t^2 = -\omega_p^2 x$$

with

$$\omega_p = \left(\frac{ne^2}{\epsilon_0 m_e}\right)^{1/2} \qquad \text{(plasma (angular) frequency).} \qquad (2.25)$$

This is the motion of a harmonic oscillator of (angular) frequency $\omega_p$.

Charge separation in a plasma therefore makes electrons oscillate at the (angular) frequency $\omega_p$. Ions, being much heavier, would oscillate more slowly (by the factor $(m_i/m_e)^{1/2}$), so that at the scale of the frequency $\omega_p$, they barely move. Numerically, the *plasma frequency* is

$$f_p = \frac{1}{2\pi}\left(\frac{ne^2}{\epsilon_0 m_e}\right)^{1/2} \simeq 9\sqrt{n} \qquad \text{plasma frequency} \qquad (2.26)$$

in SI units, i.e. with $f_p$ in Hz and $n$ in $\mathrm{m}^{-3}$. In the solar wind at 1 AU from the Sun, we have $f_p \sim 2 \times 10^4$ Hz; we shall see a direct illustration of the plasma frequency in Section 6.4. In the Earth's ionosphere (see Section 7.1), the plasma frequency is a few $10^6$ Hz; electromagnetic waves of higher frequency are reflected, a property which enabled early long-distance radio communications.

This oscillating behaviour holds under two conditions. First, the collisions between particles should not suppress the plasma oscillations. This requires the electron collision frequency to be much smaller than the plasma frequency. If the (gas) plasma is nearly completely ionised, this condition is always met since $\omega_p \sim v_{the}/L_D$, so that we have approximately

$$\frac{\nu_{ei}}{\omega_p} \sim \frac{L_D}{l_f} \simeq \Gamma^{3/2} \times \ln(1/\Gamma) \tag{2.27}$$

which is much smaller than unity in a gaseous plasma ($\Gamma \ll 1$). On the other hand, in a weakly ionised plasma, the collective behaviour requires that the collision frequency of electrons with neutrals be smaller than the plasma frequency, which requires the degree of ionisation to be large enough.

The second hypothesis made is that electrons move as a whole, i.e. that their random agitation is negligible. During a period of plasma oscillation $\sim 1/\omega_p$, the agitation displaces an electron by a distance equal to the most probable speed $\sim (k_B T/m_e)^{1/2}$ times $1/\omega_p$, i.e. the distance $L_D$. Hence, the plasma bulk oscillations occur only at scales much greater than the Debye length. It is only at such scales that a large number of particles can contribute cumulatively to produce a collective behaviour. We shall see in Section 2.3 that the electron random motion makes the plasma oscillations propagate as plasma waves, and also damps them if the scale becomes comparable with (or smaller than) the Debye length.

The origin of that collective behaviour is very different from the one in a neutral gas. In a neutral gas, the coupling between particles is due to their mutual collisions. In a plasma, the coupling is due to the mean electric field produced by particles. The collective behaviour therefore requires that the close encounters yielding large perturbations to this mean field be rare enough.

### 2.1.5 Non-classical plasmas

#### Quantum degeneracy

The above estimates are based on the assumption that the plasma behaves classically. If the concentration of particles is too high, however, their distance may involve scales so small that quantum effects act. Basically this is because, from Heisenberg's uncertainty relations, localising the particles in a small region $\Delta x$ gives them a momentum $\Delta p \sim \hbar/\Delta x$. If the density is high, then $\Delta x$ is small, and the corresponding $\Delta p$ yields a high energy.

Let us estimate this effect. Pauli's exclusion principle tells us that two plasma particles (which are fermions) cannot be in the same quantum state; hence each particle must be localised within a region of size smaller than about half the average interparticle distance, i.e. $\Delta x \sim n^{-1/3}/2$. A compression at density $n$ therefore produces the momentum $p \sim \hbar/\Delta x \sim 2\hbar n^{1/3}$ per particle. The corresponding energy is $p^2/(2m) \sim 2\hbar^2 n^{2/3}/m$ per (non-relativistic) particle of mass $m$. Because of the small electron mass, this energy is much greater for electrons than for ions.

Hence, compressing a plasma gives to each electron an energy of about $2\hbar^2 n^{2/3}/m_e$, where $n$ is their number density. A detailed calculation confirms this order of magnitude estimate; the exact values are $p_F = \hbar\left(3\pi^2 n\right)^{1/3}$ for the momentum, whence the so-called *Fermi energy* $W_F = p_F^2/(2m_e)$:

$$W_F = \hbar^2(3\pi^2 n)^{2/3}/(2m_e). \tag{2.28}$$

The greater the particle density, the smaller the region that is available to an electron, and the higher the resulting Fermi energy. If the Fermi energy becomes greater than $k_BT$, then the total energy is determined by the Fermi energy instead of $k_BT$, and the electrons are said to be *degenerate*. This occurs if the temperature is smaller than $W_F/k_B \equiv T_F$, the so-called *Fermi temperature*. In that case, the Fermi temperature (instead of the kinetic temperature) determines the particle pressure, the coupling parameter $\Gamma$ and the Debye length, so that these quantities become independent of the kinetic temperature, and only depend on the density.

We shall not encounter degenerate plasmas in this book, except when determining the limits of stellar (see Section 3.1) and planetary masses (see Section 7.1), which involve the Fermi energy.

**Relativistic particles**

Finally, to determine whether the particles are relativistic, we have to compare the average kinetic energy $\sim k_BT$ (or $k_BT_F$ if they are degenerate) with the rest mass energy $mc^2$.

For relativistic particles, the Fermi energy and temperature must be calculated with the relativistic energy–momentum relation. In particular if the particles are ultra-relativistic ($v \simeq c$), the energy of a particle of momentum $p$ is $W \simeq pc$ (instead of $p^2/2m$), so that the electron Fermi energy is now $W_F \simeq p_F c$ (instead of $p_F^2/(2m_e)$). The Fermi energy of ultra-relativistic particles therefore varies as $n^{1/3}$ instead of $n^{2/3}$.

Likewise, the pressure of ultra-relativistic particles is 1/3 of their energy density (instead of the factor 2/3 relevant in the non-relativistic case). We shall not encounter relativistic plasmas in this book (but only individual relativistic particles), but we shall use the pressure of photons (which are par excellence relativistic particles) when studying the solar interior (Section 3.1) and the dynamics of heliospheric dust grains (Section 7.4).

### 2.1.6 Summary

Gaseous plasmas have $\Gamma \ll 1$, i.e. the (Coulomb) interaction energy of the particles is much smaller than the kinetic energy. The slow decrease with distance of the Coulomb force has two major consequences. First, any particle interacts simultaneously with a large number of particles and modifies the medium so that each particle may be regarded as being 'dressed' by the other particles. This dressing makes plasmas quasi-neutral on large spatial ($L > L_D$) and temporal ($t > \omega_p^{-1}$) scales, and produces a collective behaviour. Second, the particle collisional free path increases strongly with speed, so that fast particles tend to be nearly collisionless.

## 2.2 Dynamics of a charged particle

In this section, we consider briefly the dynamics of a charged particle in electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields which are given a priori, i.e. which are negligibly modified by the moving charge itself. Most of the results will therefore be applicable to a small minority of particles which do not affect the bulk of the plasma, for example cosmic rays. The coupling between the magnetic field and the bulk plasma will be considered later.

Magnetic fields are ubiquitous in the Universe, and we shall focus on them. But why is this so? Indeed, relativity theory tells us that electric and magnetic fields are symmetrical in that they transform into each other upon a change of reference frame.

### 2.2.1 The key role of the magnetic field

As we shall see below, the key role of the magnetic field stems from two facts:

- plasmas (made of electric charges) are ubiquitous in the Universe, whereas magnetic charges (the so-called *magnetic monopoles*) are absent,
- one generally considers non-relativistic ($V \ll c$) changes of reference frames.

The absence of magnetic monopoles[7] – whereas electric charges are ubiquitous – is at the origin of the asymmetry in Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho_e/\epsilon_0 \qquad \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{2.29}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \left(1/c^2\right) \partial \mathbf{E}/\partial t \tag{2.30}$$

which contain electric charges ($\rho_e$) and currents ($\mathbf{J}$), but no magnetic charges and currents.

We have seen that plasmas are quasi-neutral on large-scales, so that the large-scale electric field nearly vanishes in the reference frame where the plasma is at rest. On the other hand, positive and negative electric charges moving differently yield electric currents, which produce magnetic fields.

Consider a plasma of (non-relativistic) bulk velocity $\mathbf{V}$ with respect to a 'laboratory' frame $\mathcal{R}$, where the electric and magnetic fields are respectively $\mathbf{E}$ and $\mathbf{B}$. The fields in the plasma frame $\mathcal{R}'$ are given by the Lorentz transformations as

$$\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B} \qquad \mathbf{B}' = \mathbf{B} - \mathbf{V} \times \mathbf{E}/c^2 \tag{2.31}$$

where we have neglected terms of order $V^2/c^2$. Since we have $\mathbf{E}' = 0$ in the

[7] You would get a magnetic charge if you could separate the two poles of a bar magnet. If these magnetic monopoles do exist, they have not yet been detected, which sets an upper limit on their concentration; see for example [13].

plasma frame $\mathcal{R}'$, the fields in the frame $\mathcal{R}$ are from (2.31)

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} \qquad \mathbf{B} = \mathbf{B}' + \mathbf{V}\times(\mathbf{V}\times\mathbf{B})/c^2 \simeq \mathbf{B}', \tag{2.32}$$

again neglecting terms of order $V^2/c^2$.[8]

Hence the magnetic field plays a privileged role: it is independent of the reference frame; in contrast, the electric field depends on the reference frame and is nearly zero in the plasma frame.[9] The latter therefore appears as a natural reference frame, and the Faraday concept of magnetic field lines acquires a basic physical meaning since if two points are connected by a field line in this frame, they are so connected in another reference frame (for non-relativistic Lorentz transformations). Magnetic field lines – the pillars of magnetohydrodynamics – have still more interesting properties, which we shall study in the next section.

Another basic property of $\mathbf{B}$ is that it is a *pseudo-vector*, since its sense depends on the usual convention of right-handed co-ordinate systems. If one changes the co-ordinate system according to $\mathbf{x} \rightarrow \mathbf{x}' = -\mathbf{x}$ (a reflection about the origin, making the co-ordinate system left-handed), the components of true vectors (as a velocity or a force) transform as $v'_x = -v_x$, leaving the physical direction of the vectors unchanged. The Lorentz force $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$ is also a true vector, so that the inversion of the co-ordinate changes the components of both $\mathbf{F}$ and $\mathbf{v}$; therefore it does not change the components of $\mathbf{B}$, whose physical direction is thus reversed. Formally, the magnetic field is analogous to a vortex. *Mirror asymmetry* plays a key role in magnetic field generation, and we shall encounter applications of this property in Sections 3.3 and 4.2.

## 2.2.2 Basic charge motion in constant and uniform fields

The basic equation of motion for a particle of charge $q$ and velocity $\mathbf{v}$ subjected to the fields $\mathbf{E}$ and $\mathbf{B}$ is

$$\frac{d(m\mathbf{v})}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{2.33}$$

with the relativistic mass

$$m = \gamma m_0 \qquad \gamma = \left(1 - v^2/c^2\right)^{-1} \tag{2.34}$$

where $m_0$ is the particle rest mass and $\gamma$ the Lorentz factor.

### Uniform magnetic field

If $\mathbf{E} = 0$ and $\mathbf{B}$ is constant, the Lorentz force reduces to $q\mathbf{v}\times\mathbf{B}$, perpendicular to the velocity; it produces a curvature of the particle path, but no change in

[8] We shall sometimes consider particles moving individually at relativistic velocities, but we shall not consider reference frames moving at relativistic velocities with respect to the bulk plasma.

[9] By 'nearly zero', we mean of amplitude small with respect to $|\mathbf{V}\times\mathbf{B}|$, and on a large scale.

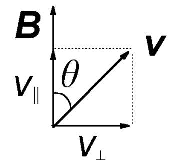


Figure 2.7 Components of the velocity parallel and perpendicular to the magnetic field; $v_\perp = v \sin\theta$, where $\theta$ is the so-called *pitch angle*.

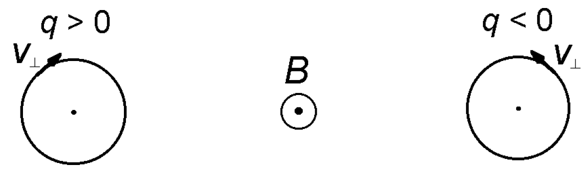


Figure 2.8 Gyration of a charge in a magnetic field (pointing out of the paper).

the speed $v$ and thus in the relativistic mass $m$. Therefore in this case the motion of a relativistic particle is the same as that of a non-relativistic particle of (constant) mass $m = \gamma m_0$. Since the force vanishes along $\mathbf{B}$, $v_\parallel$ is constant; since $v$ is constant too, so is the angle $\theta$ between $\mathbf{v}$ and $\mathbf{B}$ (Fig. 2.7). On the other hand, in the plane $\perp$ $\mathbf{B}$, the force produces a circular motion of radius $r_g$ and (angular) frequency $\omega_g = v_\perp / r_g$, given by equating the acceleration $qv_\perp B/m$ to the centrifugal acceleration $v_\perp^2/r_g$, so that

$$r_g = \frac{m v_\perp}{|\, q \,|\, B} \quad \text{(Larmor radius)} \tag{2.35}$$

$$\omega_g = \frac{|\, q \,|\, B}{m} \quad \text{((angular) gyrofrequency)} \tag{2.36}$$

with particles of negative (positive) charge gyrating in the direct (opposite to direct) sense. Hence the magnetic field generated by the particle is opposite to the imposed field (Fig. 2.8): the plasma is diamagnetic.

The *Larmor radius* (or *radius of gyration*) and the *gyrofrequency* (or *cyclotron frequency*) set the scales below which the individual particle gyration plays an important role. Numerically, the cyclotron frequency is $f_g = \omega_g/2\pi \simeq 2.8 \times 10^{10} B$ in SI units for electrons (and smaller by the factor $m_e/m_p$ for protons). In the Earth's environment, the Larmor radius is about a few centimetres for electrons and 1 m for protons; it is greater by more than five orders of magnitude in the solar wind.

The resulting path is a helix of constant pitch around a magnetic line of force. Since particles having the same value of $mv/q$ and pitch angle have the same trajectory, high-energy particles (see Section 8.2) are often quantified by their so-called *rigidity* defined as $pc/\mid q \mid$ (with $p = mv$), expressed in volts since it has the dimension of energy per charge.

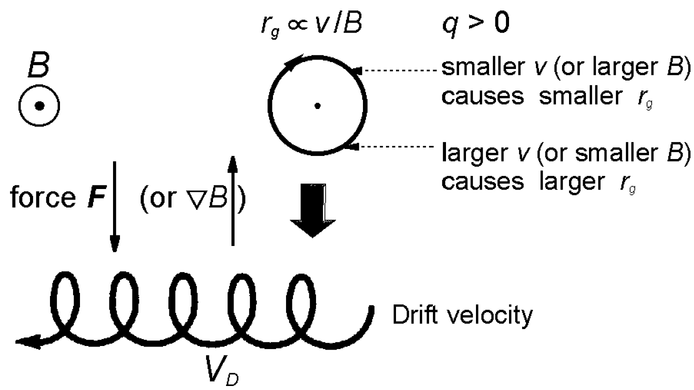


Figure 2.9 A force $\mathbf{F} \perp \mathbf{B}$ accelerates the particles along one half of the orbit and decelerates them along the other half. This makes the Larmor radius ($r_g \propto v/B$) greater near the bottom of the orbit than near the top (when $\mathbf{F}$ is downwards), which deforms the orbit, producing a drift. A gradient of $B$ in the direction $\perp \mathbf{B}$ has a similar effect.

### Electric field or applied force

How is this trajectory changed if the electric field does not vanish? Let $\mathbf{E}_\perp$ be the electric field in the direction $\perp \mathbf{B}$ in some frame $\mathcal{R}$. Consider now the reference frame $\mathcal{R}'$ moving at velocity

$$\mathbf{V}_D = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2.37}$$

with respect to $\mathcal{R}$. In $\mathcal{R}'$ the electric field $\perp \mathbf{B}$ is $\mathbf{E}'_\perp = \mathbf{E}_\perp + \mathbf{V}_D \times \mathbf{B} = 0$, since from (2.37) $\mathbf{V}_D \times \mathbf{B} = -\mathbf{E}_\perp$. Hence the motion in the plane $\perp \mathbf{B}$ reduces to the gyration found above. Going back to the frame $\mathcal{R}$, the motion in the plane $\perp \mathbf{B}$ is therefore the superposition of the gyration found above and a drift of velocity $\mathbf{V}_D$ given by (2.37). This velocity is the same for all charged particles, making the plasma move as a whole.

This drift velocity may be interpreted in either of two ways. The first way is that it produces a Lorentz force $q\mathbf{V}_D \times \mathbf{B}$ which balances the electric force $q\mathbf{E}_\perp$, so that for an observer moving at $\mathbf{V}_D$ the electric field has been transformed away. The other interpretation is sketched in Fig. 2.9. The force $q\mathbf{E}_\perp$ accelerates the particle during the part of the circular orbit where it moves in the same sense as the force, and decelerates it when it moves the other way. Hence the particle gyrates faster (thus with a greater Larmor radius) near the bottom of the orbit than near the top (when the force is downwards), producing a drift to the left when the gyration is clockwise; reversing either $\mathbf{B}$, the force or $q$ reverses the drift.

This result can be applied to other forces by replacing in (2.37) $q\mathbf{E}$ by a general force $\mathbf{F}$, which therefore produces a drift

$$\mathbf{V}_D = (\mathbf{F}/q) \times \mathbf{B}/B^2. \tag{2.38}$$

### 2.2.3 Non-uniform magnetic field

The magnetic field is generally non-uniform. If the non-uniformity is weak, namely if the field does not change much over a distance equal to the gyroradius (or during a time equal to the inverse of the gyrofrequency), the motion can be approximated by the gyration found above, around a point which is moving. This instantaneous centre of gyration is called the *guiding centre* of the particle. We calculate below its motion by considering separately the variation in magnetic field strength perpendicular and parallel to the magnetic field direction.

#### Drift produced by a variation of $B \perp \mathbf{B}$

If the magnetic field strength varies in the direction $\perp \mathbf{B}$, the magnetic field lines are curved, which forces the particles to follow curved paths along $\mathbf{B}$. If the radius of curvature of the field line is $R_c$, the centrifugal force on a particle of parallel velocity $v_\parallel$ is $F = mv_\parallel^2/R_c$, pointing opposite to the centre of curvature. This effective force produces a drift velocity given by (2.38). The particle gyration produces an additional drift because the gradient in $B$ causes the Larmor radius ($r_g \propto 1/B$) to be larger during one half of the orbit than during the other half, which deforms the orbit. This has a similar effect as an applied force (Fig. 2.9) and produces an additional drift velocity. If the magnetic field is essentially produced by exterior currents, we have $\nabla \times \mathbf{B} = 0$, whence $1/R_c = | \nabla_\perp B | /B$, where the symbol $\nabla_\perp$ denotes the component of the gradient in the direction $\perp \mathbf{B}$, and $\nabla_\perp B$ points towards the centre of curvature. Finally one finds a total drift velocity equal to

$$\mathbf{V}_D = \left( \frac{mv_\perp^2}{2} + mv_\parallel^2 \right) \frac{\mathbf{B} \times \nabla_\perp B}{qB^3}. \tag{2.39}$$

For a particle of energy $W$, we have in order of magnitude $V_D \sim W \mid \mathbf{B} \times \nabla_\perp B \mid /qB^3 \sim W/(qBR_c)$. Electrons and ions drift in opposite senses, producing an electric current.

#### Variation of $B \parallel \mathbf{B}$

Consider now a magnetic field oriented primarily along $\mathbf{z}$ with approximate cylindrical symmetry, and whose strength varies along $\mathbf{B}$. Let us assume for example $dB/dz > 0$ (Fig. 2.10). Magnetic flux tubes, whose surface is everywhere parallel to $\mathbf{B}$, have approximate cylindrical symmetry, and since the magnetic flux is a constant along a flux tube (because $\nabla \cdot \mathbf{B} = 0$), the field lines converge towards positive $z$, i.e. the radial component $B_r < 0$ . Hence the Lorentz force has a component along $\mathbf{z}$

$$F_z = \mid qv_\perp \mid B_r \tag{2.40}$$

which has the same sign as $B_r$ (here negative) whatever the sign of $q$ (since the gyration speed $v_\perp$ changes of sense as $q$ changes of sign).

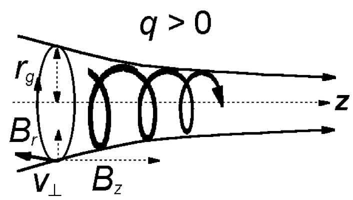


Figure 2.10 When the magnetic field strength varies along **B**, the Lorentz force on a gyrating particle has a component that decelerates (accelerates) it when it moves towards increasing (decreasing) $B$.

Hence, the motion is slowed down if the charge moves towards stronger $B$, and is accelerated if the charge moves towards smaller $B$. From $\nabla \cdot \mathbf{B} = 0$, we have $B_r = -(r/2)\, dB/dz$ at distance $r$,[10] which we substitute into (2.40) with $r = r_g = mv_\perp/qB$ to yield the force $\parallel \mathbf{B}$

$$F_\parallel = -\mu \nabla_\parallel B \tag{2.41}$$

where the symbol $\nabla_\parallel$ denotes the component of the gradient in the direction of **B**, and

$$\mu = \frac{mv_\perp^2/2}{B} \quad \text{(magnetic moment).} \tag{2.42}$$

**Magnetic moment**

The force (2.41) is the usual force on a small diamagnetic magnet lying in a gradient of magnetic field strength. Similarly, the drift velocity produced by a gradient of $B$ in the direction $\perp \mathbf{B}$ (ignoring the effect of curvature) corresponds to a force that may be written from (2.38)–(2.39) as $F_\perp = -\mu \nabla_\perp B$.

The quantity $\mu$ is called the *magnetic moment* of the particle. Indeed, the gyration of the charge averaged over one gyration corresponds to an electric current $I = \mid q \mid / (2\pi/\omega_g)$; since the loop area is $s = \pi r_g^2$ and $\omega_g r_g^2 = mv_\perp^2/qB$, the magnetic moment (2.42) is equal to $\mu = I \times s$; furthermore, since opposite charges gyrate in opposite senses, the sense of the current is independent of the sign of $q$. Hence, in average over one rotation, the particle gyration is equivalent to a current loop of magnetic moment $\mu$ given by (2.42) and pointing always opposite to **B** (Fig. 2.11). This illustrates the already mentioned plasma diamagnetism.

The potential energy of a magnetic dipole in a magnetic field **B** is $-\boldsymbol{\mu} \cdot \mathbf{B}$, and the corresponding force is formally $\nabla(\boldsymbol{\mu} \cdot \mathbf{B}) = -\nabla(\mu B)$ since $\mu$ points opposite to **B**; hence the result found above that the force is $-\mu \nabla B$ in a (weak) gradient of magnetic strength suggests that $\mu =$ constant. In fact, one may prove

[10] To prove this, draw a cylinder of radius $r$ and length $dz$ along the $z$ axis. The outward magnetic flux crossing its bounded surface is $(2\pi r dz)\, B_r + \pi r^2 \left[B_z(z+dz) - B_z(z)\right]$, which is equal to zero since $\nabla \cdot \mathbf{B} = 0$.

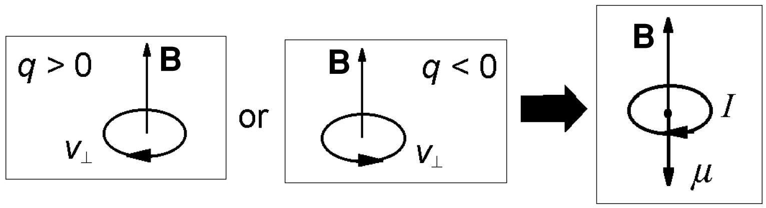


Figure 2.11 Magnetic moment produced by the gyration of a charge in a magnetic field. Whatever the sign of the charge, the current has the same direction and is equivalent (in average over one gyration) to a magnetic moment opposite to the imposed magnetic field.

a stronger result: the magnetic moment of a particle gyrating in a magnetic field remains nearly constant in both space and time when the magnetic field varies slowly at the scale of the gyration. Note that, for a relativistic particle, the conserved quantity is instead the magnetic flux $Br_g^2$ across the loop, so that the invariant is the quantity $\gamma\mu$ rather than $\mu$. This invariance is an example of Lenz's law: electrical circuits change their currents in order to counteract externally caused changes of the enclosed magnetic fluxes.

### Magnetic mirrors

This has an important consequence. When a particle is moving towards increasing magnetic field strength (converging magnetic field lines), the Lorentz force slows down the motion along $\mathbf{B}$. The perpendicular energy $mv^2 \sin^2 \theta$ increases with $B$, keeping $\mu$ constant; since $v$ remains constant because energy is conserved, $\theta$ increases, until $\theta = \pi/2$; at this point the particle is reflected back towards the weaker field. A region of increasing magnetic field thus acts as a mirror for charged particles.

Particles may therefore be trapped between two regions of strong magnetic field. This occurs close to magnetised planets having a dipolar magnetic field, where the increasing magnetic field strength in both hemispheres mirrors particles (see Appendix and Problem 2.5.3). Such particles may be viewed as small magnets (of magnetic moment pointing locally opposite to $\mathbf{B}$), which are repelled by the large 'magnet' responsible of the planetary magnetic field. When approaching the planet's positive pole (with their own positive pole pointing ahead) they are repelled back towards the planet's negative pole. Since they approach it with their negative pole ahead, they are again repelled, and keep on oscillating between the poles.

So the particles not only gyrate around field lines, but also bounce between regions of high magnetic field. Furthermore, the transverse gradient and curvature of the field lines produces a drift velocity given by (2.39). With a dipolar magnetic field, $\mathbf{B} \times \nabla_\perp B$ is in the azimuthal direction, making the particles drift in longitude (Fig. 2.12).

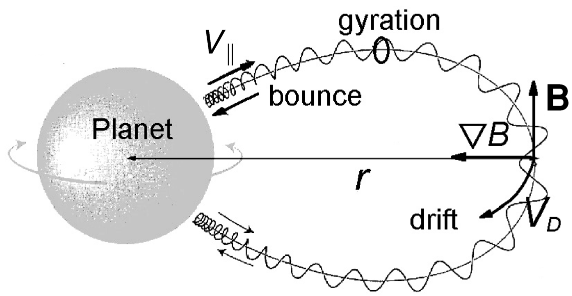


Figure 2.12 Charged particles can be trapped in the *magnetic bottle* formed by a dipolar planetary magnetic field. Their motion is the superposition of three components: the *gyration* around a field line, the *bounce* between magnetic mirrors in opposite hemispheres, and an azimuthal *drift* produced by the transverse magnetic field gradient.

### 2.2.4 Adiabatic invariants

The near invariance of $\mu$ is a particular case of adiabatic invariance, associated to periodic generalised co-ordinates of Hamiltonian systems [12]. When a parameter varies slowly at the scale of the period, the action integral varies much less than this parameter, and is called an *adiabatic invariant* – by analogy with thermodynamics where ~~adiabatic processes are generally slow~~. A classical example is the oscillating pendulum, whose adiabatic invariant is the energy divided by the frequency; indeed, if one changes slowly the length of a pendulum, the frequency varies in proportion to the energy. The same holds for a particle gyrating in a magnetic field. Since the particle motion has three degrees of freedom, there may be three adiabatic invariants if the system has several periodicities.

In this way, a particle trapped in a dipolar magnetic field has three adiabatic invariants associated to the three periodic motions:

- the gyration around magnetic field lines (speed $v_\perp$, period $T_1 \sim r_g/v_\perp$), whose adiabatic invariant is $\mu$, given by (2.42),[11]
- the bounce between mirror points (speed $v_\parallel$, period $T_2 \sim r/v_\parallel \gg T_1$, where $r$ is the distance to the planet), whose adiabatic invariant is the integral of the longitudinal momentum $mv_\parallel$ along the path between mirror points,
- the azimuthal drift produced by the gradient in magnetic field strength perpendicular to $\mathbf{B}$ (speed $V_D$ given by (2.39), whence in order of magnitude $V_D \sim mv_\perp^2/(qBr) \sim v_\perp r_g/r$, period $T_3 \sim r/V_D \gg T_2$), whose adiabatic invariant is the magnetic flux across the area encircled by the drift path.

[11] Or rather $\gamma\mu$, if the particles are relativistic.

This adiabatic invariance has a number a consequences. It enables particles to remain trapped for a long time on the same magnetic shell in dipolar planetary magnetic fields; this is responsible for the long life of radiation belts around planets. We shall apply these concepts to the trapping and acceleration of particles in the contexts of magnetospheres (Section 7.3) and cosmic rays (Section 8.2).

### 2.2.5 Summary

Charged particles gyrate around magnetic field lines, keeping their magnetic moment invariant when the magnetic field varies weakly at the scale of the gyration. A weak longitudinal increase in magnetic field strength acts as a magnetic mirror. A force, or a weak transverse magnetic gradient, produces a small transverse drift.

## 2.3 Many particles: from kinetics to magnetohydrodynamics

A plasma is made of a large number of particles. In classical mechanics, the state is defined by the position $\mathbf{r}$ and velocity $\mathbf{v}$ of each particle at time $t$, and the evolution is determined by the equation of motion of each particle. To make the problem tractable, one has to decrease the number of variables. This is done by making averages, in two main ways:

- the *kinetic description* retains some microscopic properties by considering as the basic quantity the velocity distribution (for each particle species); basically, this amounts to replacing the equations of motion for each particle by a differential equation on the velocity distribution,

- the *fluid description* deals with a few macroscopic quantities as the mean density of particles (or of mass), the mean velocity, the pressure or the temperature, etc., which represent averages over the velocity distribution (for each particle species); basically, this amounts to replacing the velocity distribution – a function generally defined by an infinite number of parameters – by a few parameters, which is permissible only near thermodynamic equilibrium.

### 2.3.1 Elements of plasma kinetics

We define the particle velocity distribution[12] so that the number of particles in the volume element $[x, x + dx]$,$[y, y + dy]$,$[z, z + dz]$, and with velocities in the range $[v_x, v_x + dv_x]$,$[v_y, v_y + dv_y]$, $[v_z, v_z + dv_z]$ at time $t$ is

$$d^6N = f(\mathbf{r}, \mathbf{v}, t) \times d^3r \times d^3v \tag{2.43}$$

[12]Beware that there are many subtleties in this definition, as discussed for example in [6].

where

$$d^3r = dxdydz \tag{2.44}$$
$$d^3v = dv_x dv_y dv_z \tag{2.45}$$

are the volumes in the space of positions and the space of velocities, respectively. A position $\mathbf{r}$ and velocity $\mathbf{v}$ thus correspond to a 'point' in a phase space of six dimensions $[x, y, z, v_x, v_y, v_z]$, which we denote $[\mathbf{r}, \mathbf{v}]$.[13]

This description is more complete than the fluid description which deals with averages of $f$ as

$$n = \int d^3v \, f(\mathbf{v}) \qquad \text{(particle density)} \tag{2.46}$$

$$\mathbf{V} \equiv \langle \mathbf{v} \rangle = \int d^3v \, f(\mathbf{v}) \times \mathbf{v}/n \qquad \text{(velocity)} \tag{2.47}$$

and higher-order moments, for each species of particles; we have not written explicitly the dependence in $\mathbf{r}$ and $t$, to simplify the notations. These moments are macroscopic quantities defined in the ordinary space of three dimensions $[x, y, z]$.

### A reminder: the Maxwellian distribution

In the special case of thermodynamic equilibrium at temperature $T$, statistical mechanics tells us that, as we already noted, the velocity is Gaussian distributed along each co-ordinate (in the frame where the mean velocity vanishes), as

$$\begin{aligned} f(\mathbf{v}) &= Ae^{-mv_x^2/(2k_BT)} \times e^{-mv_y^2/(2k_BT)} \times e^{-mv_z^2/(2k_BT)} \\ &= A\exp[-mv^2/(2k_BT)] \end{aligned} \tag{2.48}$$

for particles of mass $m$, where $A = n\,[m/(2\pi k_BT)]^{3/2}$ to ensure the normalisation (2.46). This is the Maxwell–Boltzmann distribution.

Beware that the probability for the speed $v = \left(v_x^2 + v_y^2 + v_z^2\right)^{1/2}$ to lie in the range $[v, v + dv]$ is not $f(\mathbf{v})\,dv$ but

$$f(\mathbf{v}) \times 4\pi v^2 dv \tag{2.49}$$

since this range corresponds to a volume of velocity space equal to that of a spherical shell of radius $v$ and width $dv$, that is $d^3v = 4\pi v^2 dv$. Throughout this book, the notation $f(\mathbf{v})$ (which reduces to $f(v)$ when the distribution is isotropic) denotes the distribution defined by (2.43).

With the Maxwellian distribution (2.48), the distribution in speeds ($v$) thus varies as $v^2 e^{-mv^2/(2k_BT)}$, so that the most probable speed (the one at which the derivative of $v^2 e^{-mv^2/(2k_BT)}$ vanishes) is

$$v_{th} = (2k_BT/m)^{1/2}\,. \tag{2.50}$$

[13] We consider below non-relativistic motions. In the relativistic case, one represents $f$ in terms of $\mathbf{r}$ and $\mathbf{p} = m\mathbf{v}$ instead of $\mathbf{r}$ and $\mathbf{v}$.

On the other hand, the mean square speed[14] is $\langle v^2 \rangle = 3k_B T/m$, so that, as we already noted, the average kinetic energy per particle is

$$W = m\langle v^2 \rangle /2 = 3k_B T/2. \tag{2.51}$$

### Evolution of *f*

From the definition (2.43), the velocity distribution $f$ is a density in the (six-dimensional) phase space $[\mathbf{r}, \mathbf{v}]$, just as $n$ (or $\rho$) is the particle (or mass) density in the ordinary (three-dimensional) space of positions. Let us study how $f$ evolves.

As $t$ varies, the distribution $f$ varies, while the position $\mathbf{r}$ and the velocity $\mathbf{v}$ of a particle vary as

$$\mathbf{dr} = \mathbf{v}dt \tag{2.52}$$

$$\mathbf{dv} = \mathbf{a}dt \tag{2.53}$$

where the acceleration vector is from the equation of motion (2.33)

$$\mathbf{a} = \mathbf{dv}/dt = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)/m \tag{2.54}$$

for particles of charge $q$, mass $m$ and velocity $\mathbf{v}$, in the fields $\mathbf{E}$ and $\mathbf{B}$, to which must be added in general a gravitational acceleration.

Therefore, the evolution of $f$ may be seen from two different points of view: the variation with time at a fixed position and velocity (the so-called *Eulerian* point of view), and the variation following particles in their motion (the so-called *Lagrangian* point of view). In the latter viewpoint, the variation has two origins: the time variation proper (at fixed co-ordinates $[\mathbf{r}, \mathbf{v}]$), and the variation of the co-ordinates $[\mathbf{r}, \mathbf{v}]$ themselves.

### The convective derivative

A similar distinction holds in fluid mechanics. Assume for example that you wish to analyse the composition of water in a river. You may do so in two ways. You may sit on the bank, and so observe the time evolution at a fixed position; by convention, we note the variation so observed as $\partial/\partial t$. A second method is to embark on a boat that drifts following the river motion; the observed variation is then noted $d/dt$. Both variations are related in one dimension ($x$) by the fact that a quantity $n$ is a function of $x$ and $t$: $n = n(x,t)$ with $x = x_0 + V_x t$ ($V_x$ being the fluid velocity). Hence $dn/dt = \partial n/\partial t + \partial n/\partial x \times dx/dt = \partial n/\partial t + V_x \partial n/\partial x$. In three dimensions, the time variation as observed following a fluid moving at velocity $\mathbf{V}$ is therefore

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \bigtriangledown) \qquad \text{(convective derivative)} \tag{2.55}$$

with the usual notation $\mathbf{V} \cdot \bigtriangledown = V_x \partial/\partial x + V_y \partial/\partial y + V_z \partial/\partial z$.

[14] Defined as $\langle v^2 \rangle = \int d^3v \; v^2 \; f(\mathbf{v})/n$.

**Vlasov equation**

A basic result of statistical mechanics is *Liouville's theorem*, a consequence of which is that *in the absence of collisions*, $f$ is invariant following the motion in the six-dimensional phase space. In other words, $df/dt = 0$, where the derivative must be understood as a convective derivative *in the six-dimensional phase space*. Just as the convective derivative in ordinary three-dimensional space $[\mathbf{r}]$ is given by (2.55), the convective derivative in the six-dimensional phase space $[\mathbf{r}, \mathbf{v}]$ is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}}. \tag{2.56}$$

Substituting (2.56), $df/dt = 0$ may be written

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \qquad \text{(Vlasov equation)}. \tag{2.57}$$

Here the acceleration $\mathbf{a}$ is given in (2.54), where in the general case, the electric and magnetic field are the mean fields produced by all the plasma particles, and one must add a gravitational acceleration if it is not negligible.

This means that in the absence of collisions, the velocity distribution (the density in the six-dimensional phase space) behaves as an incompressible (six-dimensional) fluid. Note that since $\partial \mathbf{a}/\partial \mathbf{v} = 0$ (any component of $\mathbf{a}$ is independent on the velocity along the same direction because the Lorentz force is perpendicular to the velocity),[15] we have $\mathbf{a}\partial f/\partial \mathbf{v} = \partial/\partial \mathbf{v}\,(\mathbf{a}f)$, so that (2.57) is equivalent to a *continuity equation* (in the six-dimensional phase space): $\partial f/\partial t + \partial\,(\mathbf{v}f)\,/\partial \mathbf{r} + \partial\,(\mathbf{a}f)\,/\partial \mathbf{v} = 0$.

**A reminder: the continuity equation**

The most basic equation of fluid mechanics is the continuity equation, which merely states the conservation of the number of particles or of the mass. It may be derived as follows. In the absence of creation or destruction of particles, the time variation of the number of particles in a *fixed* volume $\upsilon$ is the opposite of the outward flux of particles crossing the surface $\Sigma$ bounding this volume

$$\frac{\partial}{\partial t} \int_{\upsilon} d^3 r\; n = -\int_{\Sigma} \mathbf{dS} \cdot n\mathbf{V} = -\int_{\upsilon} d^3 r\;\; \bigtriangledown \cdot (n\mathbf{V}) \tag{2.58}$$

where the second equality has been obtained by transforming the surface integral into a volume integral by Gauss's theorem. Since this is true for any arbitrary (fixed) volume $\upsilon$, we have

$$\frac{\partial n}{\partial t} + \bigtriangledown \cdot (n\mathbf{V}) = 0 \qquad \text{(continuity equation)}. \tag{2.59}$$

In the particular case when the fluid is incompressible, we have $dn/dt = 0$, so that, using (2.55), the continuity equation is equivalent to $\bigtriangledown \cdot \mathbf{V} = 0$.

[15] By $\partial \mathbf{a}/\partial \mathbf{v} = 0$, we mean $\partial a_i/\partial v_i = 0$ for $i = x, y, z$. This property also holds with a gravitational force, but not with a friction force.

### Other forms of the invariance of $f$

There are several other equivalent ways of expressing the conservation of $f$ along particle trajectories. Since the number of particles in a volume of (six-dimensional) phase space is conserved, and so is $f$, so is this volume. The total volume in phase space therefore remains constant, whatever its change of shape as the system evolves.

Another consequence of the conservation of $f$ along particle trajectories is that the motion of charges in given electric and magnetic fields (plus possibly a gravitational field) may be calculated by expressing $f$ in terms of the constants of motion (energy, magnetic moment, ...); this is often called *Jeans' theorem*.

One must be careful to apply these results within their limits of application. In particular, they do not hold in the following cases:

- when the number of particles is not conserved (for example because of ionisation or recombination),
- when the acceleration varies with the velocity as $\partial \mathbf{a}/\partial \mathbf{v} \neq 0$,
- when collisions act,
- for values of $\mathbf{r}$ and $\mathbf{v}$ that are not accessible along particle trajectories, given the constants of motion.

When collisions are not negligible, $df/dt \neq 0$, which produces a non-zero term $(\partial f/\partial t)_c$ on the right-hand side of (2.57). In neutral gases, collisions involve two-particle encounters producing large perturbations, and this yields the *Boltzmann equation*. In plasmas, collisions act through the accumulation of small-angle Coulomb encounters, and this yields the *Fokker–Planck equation*.

### Basic illustration: effect of a force on the velocity distribution

Most textbook applications of the Vlasov equation consider waves. We study here a stationary problem: the effect of a (conservative) force on the distribution of particles.

In its simplest form, the problem may be stated as follows. We know that in equilibrium at temperature $T$ the density of particles subjected to a force deriving from a potential $\psi$ is proportional to $e^{-\psi/k_B T}$ – the Boltzmann factor. How is this result changed in absence of equilibrium? This problem arises when measuring particles aboard a spacecraft, and on a larger scale (with subtle differences) when calculating the distribution of particles near a planet or a star, and the production of a wind.

Let us assume for simplicity that the particle velocity distribution is isotropic (i.e. depends only on the modulus $v$ of the velocity) at some position. An isotropic velocity distribution may be expressed as a function of the particle energy only, which is a constant of motion. To further simplify the problem, let us assume that the potential depends on one co-ordinate only, for example the distance $r$ from an object. For a given particle, the total (conserved) energy

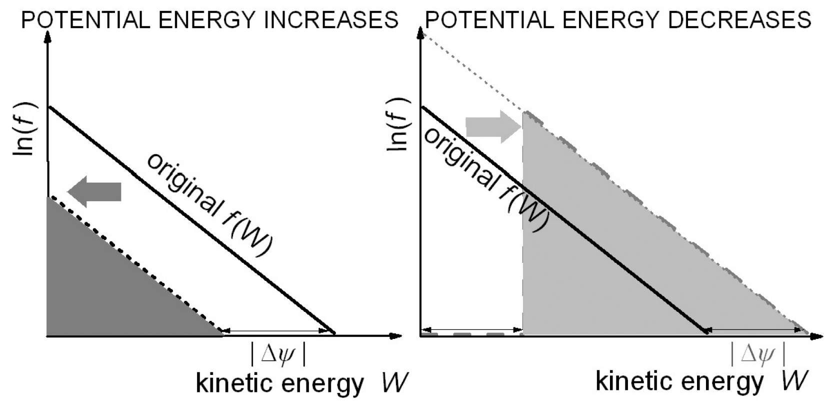


Figure 2.13 How the particle velocity distribution is modified when the potential energy increases (left) or decreases (right) by $\mid \Delta\psi \mid$, in the special case when the original distribution is a Maxwellian of temperature $T$. For $\Delta\psi > 0$ (particles coming against the force, left panel), the translation in energy decreases $f$ by the Boltzmann factor $e^{-\Delta\psi/k_B T}$. For $\Delta\psi < 0$ (particles accelerated by the force, right panel) the translation in energy produces a hole in the distribution (thick dashed grey line); if collisions did act, they would fill the hole (thin dotted grey line).

is the sum of the kinetic energy $W$ plus the potential energy $\psi(r)$. If $\psi$ varies by $\Delta\psi$ between $r_0$ and $r$, then a particle of original kinetic energy $W_0$ at $r_0$ will have at $r$ the kinetic energy $W_0 - \Delta\psi$, which is smaller or greater than $W_0$ depending on whether $\Delta\psi > 0$ (particles coming against the force) or $\Delta\psi < 0$ (particles coming in the direction of the force). From the conservation of $f$ along particle trajectories, the velocity distribution at distance $r$ (expressed as a function of the kinetic energy $W$) is thus related to the original distribution by the relation

$$f\,(r, W) \;=\; f\,(r_0, W + \Delta\psi(r)) \tag{2.60}$$

for values of $[r, W]$ accessible from $r_0$.

Therefore, the distribution $f(r, W)$ at some distance $r$ is deduced from the original distribution at $r_0$ by a translation in energy of amplitude $\Delta\psi(r)$ – the variation in potential energy. This is sketched in Fig. 2.13 in the simple case when the original distribution is a Maxwellian of temperature $T$, i.e. $\propto e^{-W/k_B T}$, so that $\ln(f)$ as a function of $W$ is originally a straight line.

The final distribution depends strongly on the sign of $\Delta\psi$. When the potential energy increases (particles coming against the force), the particle kinetic energy decreases, so that the kinetic energy distribution is translated to the left (Fig. 2.13, left). Therefore, with a Maxwellian original distribution, the particle density is reduced by the factor $e^{-\Delta\psi}$, the usual Boltzmann factor. Hence, in this case the *collisionless kinetic* description gives exactly the same result as if there were enough collisions to ensure equilibrium.

This is not so, however, when the potential energy decreases (particles moving in the direction of the force; Fig. 2.13, right). In that case, the translation in energy produces a hole in the velocity distribution, so that the density of these particles does not simply increase by the Boltzmann factor $e^{-\Delta\psi} \equiv e^{|\Delta\psi|}$ as in equilibrium. The origin of this hole is that particles of nearly zero kinetic energy at $r_0$ are accelerated to a kinetic energy equal to $\mid \Delta\psi(r) \mid$ at distance $r$, so that there is no particle of kinetic energy smaller than this value. In contrast, with collisions, the redistribution between degrees of freedom populates the hole, producing the usual Boltzmann factor.

This result illustrates the difference in Debye shielding for attracted and repelled particles in the absence of equilibrium, and has consequences on measurements of velocity distributions in space. We shall encounter a similar problem when calculating the plasma distribution in the solar corona (Section 4.6), and the solar wind acceleration (Section 5.5), with important differences: the large size of the system ensures electric quasi-neutrality; it enables collisions to populate some orbits, suppressing the hole; and the original distribution is generally not a Maxwellian.

This last point introduces a basic consequence of the rarity of collisions in space. Consider in more detail the left-hand panel of Fig. 2.13 ($\Delta\psi > 0$). The translation of the original Maxwellian distribution $f(W) \propto e^{-W/k_B T}$ yields a straight line (in log co-ordinates) having the same slope, i.e. a Maxwellian of the same temperature. But think what happens if the original distribution is not Maxwellian, a frequent situation in space. Then we no longer have a straight line, i.e. the slope (in log co-ordinates) does depend on the energy, so that the translation in energy does change the shape, thereby changing the effective temperature. Indeed, the potential filtrates the particles, letting only the fastest ones climb the potential barrier (Problem 2.5.4). This is a purely kinetic effect, completely outside the scope of the usual fluid description, and is of far-reaching consequences (Section 4.6.4).

## 2.3.2 First-aid kit for space plasma fluids

### The infinite hierarchy of fluid equations

The simplest fluid picture describes each particle species by three macroscopic quantities:

- the particle density $n$ defined by (2.46),
- the velocity $\mathbf{V} = \langle \mathbf{v} \rangle$ defined by (2.47),
- the pressure.

If the particle velocity distribution is isotropic in the frame where the mean velocity vanishes, the pressure is a scalar defined by

$$P = \frac{m}{3} \int d^3v \; f(\mathbf{v}) \left(\mathbf{v} - \langle \mathbf{v} \rangle\right)^2 \qquad \text{(pressure)} \tag{2.61}$$

for non-relativistic particles of mass $m$. The temperature is defined from $k_B T = P/n = Pm/\rho$, which, for a Maxwellian, coincides with the usual thermodynamic temperature. This generalises the definitions (2.2) and (2.3) to frames where the mean velocity does not vanish.

In this simplified picture, for a plasma containing $n$ electrons and $n$ (singly charged) ions per unit volume at the same temperature, the mass density is $\rho \simeq nm_i$ since the mass is essentially carried by the ions, but both species contribute equally to the pressure so that $P = 2nk_B T$. Hence $P = \rho k_B T/m$ where $m \simeq m_p/2$ is the average mass per particle.

With these three unknowns: $\rho$, $\mathbf{V}$ and $P$ (or equivalently, $T$, since $P = \rho k_B T/m$), three equations are required to solve the problem. We have already written the first fluid equation – the continuity equation (2.59), for the particle number density $n$; an equivalent equation holds for the mass density $\rho$. The continuity equation may be obtained more formally by integrating over the velocities the equation of evolution of $f$; elastic collisions do not change the result since they do not change the number of particles.

In a stationary case, the continuity equation means that the mass entering a flow tube (a tube everywhere parallel to the fluid velocity) across a given section equals the mass leaving across another section. This yields

$$\rho V s = \text{constant} \tag{2.62}$$

if $s$ is the cross-section area. In the particular case when the medium is incompressible ($\rho$ = constant), this means that the flow lines diverge (converge) when the speed decreases (increases), or alternatively that the flow accelerates in a constricted tube – properties that are well known in hydrodynamics. In spherical symmetry, where the velocity is radial and the flow tubes vary as the square of the distance $r$, this yields $\rho V r^2$ = constant.

Similarly, the fluid equation of motion may be derived in two ways. The most intuitive way is to note that for a fluid parcel of volume unity and mass $\rho$, the force $\rho \mathbf{d}\mathbf{V}/dt$ (following the parcel's motion) is equal to the sum of the pressure force $-\nabla P$ and the gravity force $-\rho \nabla \Phi_G$, plus electric and magnetic forces if charges and currents are not negligible. Substituting the convective derivative (2.55), this yields the fluid equation of motion

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\,\mathbf{V} = -\frac{\nabla P}{\rho} - \nabla \Phi_G \quad \text{(fluid equation of motion)} \tag{2.63}$$

(where we have omitted for the moment the electromagnetic force and also the viscosity force, which will be considered later). This equation of motion may be obtained more formally by multiplying the equation of evolution of $f$ by $\mathbf{V}$ and integrating over the velocities. With a velocity distribution that is isotropic in the frame where the mean velocity vanishes, this yields (2.63), to which must be added electric and magnetic forces if there is a finite density of charge and current.

An important problem emerges, which is perhaps one of the most difficult problems of space plasma physics. With the continuity equation and the fluid equation of motion, we have only two equations for the three unknowns $\rho$, $\mathbf{V}$ and $P$ (or equivalently, $T$, since $P = \rho k_B T/m$). One might think naively that this problem could be solved by going a step further in the averaging process. Indeed, the continuity equation stems from averaging the equation on $f$, the equation of motion stems from averaging the equation on $f$ multiplied by $\mathbf{V}$, and similarly another equation (involving the energy) stems from averaging the equation on $f$ multiplied by $\mathbf{V}^2$. Unfortunately this does not work, because just as the continuity equation determines $\rho$ in terms of $\mathbf{V}$, the equation of motion (2.63) determines $\mathbf{V}$ in terms of $\rho$ and $P$ (or $T$), a third equation will involve a moment of higher order, in terms of which $P$ (or $T$) will be expressed, and so on up to an infinite number of moments.

The fluid equations therefore constitute a ladder having an infinite number of steps. This is not surprising since one cannot replace an infinite number of unknowns (which a velocity distribution represents effectively in the general case) by a finite number of unknowns – for example $\rho$, $\mathbf{V}$ and $P$ – without a miracle. The root of the problem is the transport of energy, which we shall examine below. Meanwhile, let us examine some cases when the miracle comes true.

### Bernoulli's theorem

Let us make two assumptions:

- the problem is stationary, so that $\partial/\partial t = 0$,
- $P$ is a function of $\rho$ only (for example $P$ and $\rho$ obey a relation $P \propto \rho^\gamma$), or the fluid is incompressible.

In this case, we can define the enthalpy

$$H = \int dP/\rho \qquad \text{(enthalpy)}. \tag{2.64}$$

We deduce $\bigtriangledown P/\rho = \bigtriangledown H$, so that multiplying the equation of motion (2.63) by $\mathbf{V}$, we obtain

$$\mathbf{V} \cdot \bigtriangledown \left[V^2/2 + H + \Phi_G\right] = 0. \tag{2.65}$$

This means that we have along flow lines

$$V^2/2 + H + \Phi_G = \text{constant} \qquad \text{(Bernoulli's theorem)}. \tag{2.66}$$

Bernoulli's theorem is a pillar of fluid dynamics. It may be used to solve a host of problems, from domestic plumbing to astrophysics, including how wings of aeroplanes, insects and birds produce lifts (or crashes), how termites and prairie dogs design the ventilation of their homes,[16] or why your shower curtain engulfs you every morning.[17]

[16] See for example Vogel, S. 1998, *Cats' Paws and Catapults*, London, Penguin Books.
[17] The latter explanation is still under debate.

Let us calculate the enthalpy. Consider first an incompressible medium; from $P = \rho k_B T/m$ with $\rho =$ constant, we have $H = P/\rho = k_B T/m$.

Then consider the polytrope case $P \propto \rho^\gamma$. The case $\gamma = 1$ must be considered separately; we then have $T =$ constant, so that from $P = \rho k_B T/m$

$$H = \int dP/\rho = (k_B T/m) \ln \rho \qquad \text{(isothermal)}. \tag{2.67}$$

On the other hand, if $P \propto \rho^\gamma$ with $\gamma \neq 1$, we have $P \propto T^{\gamma/(\gamma-1)}$, so that

$$H = \int dP/\rho = \frac{\gamma}{\gamma - 1} \int \frac{P}{\rho} \frac{dT}{T} = \frac{\gamma}{\gamma - 1} \frac{k_B T}{m}. \tag{2.68}$$

With these values of $H$, the Bernoulli theorem (2.66) yields along flow lines, for a polytrope $P \propto \rho^\gamma$:

$$\frac{V^2}{2} + \frac{k_B T}{m} \ln \rho + \Phi_G = \text{constant} \qquad \gamma = 1 \tag{2.69}$$

$$\frac{V^2}{2} + \frac{\gamma}{\gamma - 1} \frac{k_B T}{m} + \Phi_G = \text{constant} \qquad \gamma \neq 1. \tag{2.70}$$

Equation (2.69) holds when transformations are so slow that *isothermal* equilibrium has enough time to establish everywhere.

On the other hand, (2.70) holds in the opposite case when transformations are so fast that heat has no time to flow: such processes are called *adiabatic*; in that case, $\gamma = c_p/c_v = 1 + 2/N$ – the ratio of specific heats for (non-relativistic) particles having $N$ space degrees of freedom, so that $\gamma = 5/3$ for $N = 3$. Bernoulli's theorem then represents the conservation of the fluid energy per unit mass, which is the sum of the bulk kinetic energy $V^2/2$, plus the thermal energy $3k_B T/2$, plus the work $k_B T$ expended on compression, plus the gravitational energy $\Phi_G$. Beware that (2.70) holds in the stationary case. In a time-dependent case, the adiabatic fluid energy equation simply reads $d\left(P\rho^{-\gamma}\right)/dt = 0$ with $P = \rho k_B T/m$ and $d/dt$ the convective derivative. With the help of the continuity equation and the equation of motion, it may be put under various (more or less complicated) forms. These three equations are known as the *Euler equations*.

The same result may be obtained formally by assuming the particle velocity distribution to be a Maxwellian of temperature $T$ centred on a mean velocity $\mathbf{V}$, and making averages of the equation of evolution of $f$ multiplied by $\mathbf{V}^2$. Since such a Maxwellian distribution is characterised by only three parameters – the density, the mean velocity and the temperature (or the pressure) – no miracle is needed to reduce its evolution to three equations involving these three unknowns. The establishment of a Maxwellian distribution, however, generally requires some process – such as collisions between particles – to ensure equilibrium, and is not so easily achieved in plasmas as in neutral gases, due to the nature of collisions; we shall return to this point later. Note that the relation $P \propto \rho^\gamma$ further reduces the number of independent unknowns to two, so that the equations of motion and of energy then become equivalent.

 

**Sound waves and their plasma counterparts**

Assume that the only force is the pressure force, and consider small perturbations around the simple solution having the velocity $\mathbf{V}_0 = 0$ and uniform mass density $\rho_0$ and temperature $T_0$ (and pressure $P_0 = \rho_0 k_B T_0/m$). Assume the perturbations to be fast enough for the behaviour to be adiabatic, i.e. $P \propto \rho^\gamma$, so that we have $\nabla P = (dP/d\rho) \nabla \rho$ with

$$dP/d\rho = \gamma P/\rho = \gamma k_B T/m. \tag{2.71}$$

Let us now write the continuity equation and the fluid equation of motion with the perturbed quantities $\mathbf{V} = \mathbf{V}_1$, $\rho = \rho_0 + \rho_1$, $P = P_0 + P_1$, where the subscript 1 denotes small perturbations of the initial solution. To first order in the perturbation, this yields

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{V}_1 = 0 \tag{2.72}$$

$$\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} = - \nabla P_1 = -\frac{dP}{d\rho} \nabla \rho_1. \tag{2.73}$$

Taking the time derivative of (2.72) and substituting (2.73), we obtain

$$\frac{\partial \rho_1}{\partial t^2} = V_S^2 \nabla^2 \rho_1 \tag{2.74}$$

$$V_S = \left(\frac{dP}{d\rho}\right)^{1/2} \tag{2.75}$$

and equations similar to (2.74) for the perturbations $P_1$ and $\mathbf{V}_1$. This has a plane wave solution varying as $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, which propagates at the speed $\omega/k = V_S$. With $P \propto \rho^\gamma$, we have from (2.71)

$$V_S = (\gamma P/\rho)^{1/2}. \tag{2.76}$$

This wave produces small perturbations in the fluid mass density and velocity; it is a *sound wave*, propagating at the *sound speed* $V_S = (\gamma k_B T/m)^{1/2}$ in a gas of particles of mass $m$ at temperature $T$. Note that from (2.73), $\mathbf{V}_1$ is parallel to $\nabla \rho_1$. In the Fourier space $[\omega, \mathbf{k}]$ (see [3]), $\partial/\partial t$ transforms into $-i\omega$ and $\nabla$ into $i\mathbf{k}$, so that (2.73) yields $-i\omega\rho_0\mathbf{V}_1 = -V_S^2 i\mathbf{k}\rho_1$, whence $\mathbf{V}_1 \parallel \mathbf{k}$. This shows that in a sound wave, the velocity perturbation is parallel to the wave vector: this is called a *longitudinal wave.*

In a plasma, two major differences arise. First, the pressure is provided by electrons and ions in proportion of their temperatures, whereas the mass is essentially provided by the ions. The corresponding wave – called *ion–acoustic wave* – behaves differently depending on whether or not the electrons and ions move together, i.e. whether or not the plasma remains neutral.

If the plasma remains neutral, which holds at scales greater than $L_D$ (i.e. wave numbers $k \ll 1/L_D$), the wave is a simple generalisation of the sound wave in a neutral gas, so that the phase speed is

$$V_S = \left(\frac{\gamma_e P_e + \gamma_i P_i}{\rho}\right)^{1/2} = \left(\frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}\right)^{1/2} \tag{2.77}$$

where the subscripts $e$ and $i$ refer to electrons and ions respectively. If $\gamma_e \sim \gamma_i \sim \gamma$ and $T_e \sim T_i \sim T$, the phase speed reduces to $(\gamma k_B T/m)^{1/2}$ (in order of magnitude) with $m \simeq m_i/2$ (the average particle mass), as for a neutral gas. In this case, however, since the wave phase speed is of the same order of magnitude as the most probable speed of ions, the bulk of the ions move together with the wave so that they are subjected to a nearly constant force along their motion; this accelerates them efficiently at the expense of the wave, which is thus damped; this process is called *Landau damping*, and we shall return to it in Section 2.3.4. For the wave not to be damped, the electrons must be much hotter than the ions, to produce a wave speed much greater than the ion most probable speed so that the wave is no longer damped.

When the wave number $k \geq 1/L_D$, the plasma does not remain neutral, i.e. the electrons and ions do not move together; in the large $k$ limit, the ions then perform plasma oscillations at their characteristic frequency

$$\omega_{pi} = \left(ne^2/\epsilon_0 m_i\right). \tag{2.78}$$

The picture is then like ordinary plasma oscillations (Section 2.1.4), but with the role of electrons and ions reversed.

The second major modification that plasmas introduce in the sound wave arises when a magnetic field is present. We shall consider this point later.

**Shocks**

Sound waves (and their plasma generalisations) enable fluids to adapt gently to compressions. Hence, for motions at a speed smaller than the sound speed, the fluid behaves as if it were roughly incompressible. On the other hand, for larger speeds, nasty things may happen. To understand this, suppose you agitate your hand. In doing so, you compress the surrounding air, and your hand is able to move because the gas ahead goes out of the way. It can do so because the compression is transmitted farther away by the sound waves emitted by your moving hand, thereby transmitting to the gas ahead the information that your hand is approaching.

But suppose you try to move it faster than the sound speed. In that case, sound waves do not propagate fast enough to transmit the information that your hand is moving. Hence the gas far ahead, being not aware of the motion, is not perturbed. In contrast, close to your hand, the gas is compressed by the motion and moves at the same speed; the (adiabatic) compression also increases the local temperature (and sound speed), so that the information propagates just ahead of your moving hand.

Therefore the gas separates into two regions. Far away, it remains undisturbed; in the frame of the moving object that is approaching at the speed $-V_1$,

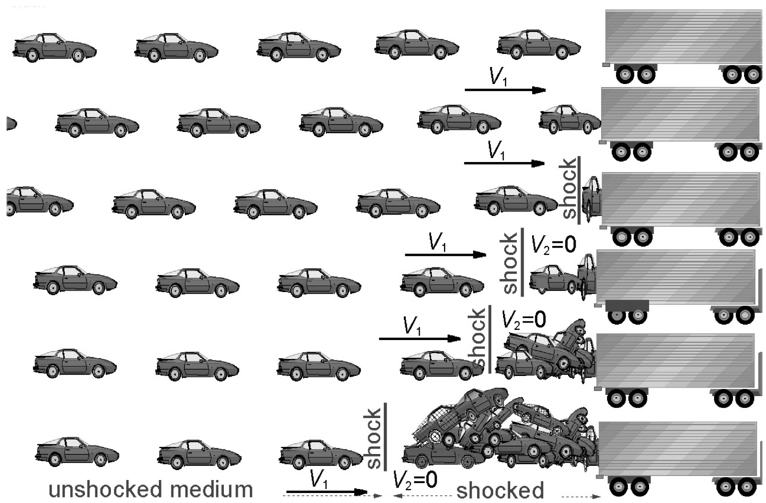


Figure 2.14 A simple shock: moving cars, whose drivers are asleep, are impacting a stopped truck. The rows show the state of the system at several consecutive times. A shock forms, separating two different states: upstream, the unperturbed fluid (the row of cars) moving at uniform velocity; downstream, the fluid that has stopped and undergone an irreversible transition. The shock moves towards the left, propagating information on the presence of the obstacle.

the gas moves at $V_1$, faster than the local sound speed. On the other hand, just before the object, the flow adapts gently, moving at a speed that is locally subsonic, and stopping at the object. In between lies a transition, at some distance ahead of the object, where the gas velocity changes from supersonic to subsonic. It is this transition that transmits the information on the presence of the obstacle ahead, and it does so at a supersonic speed – a performance that the small amplitude sound waves cannot achieve.

Contrary to the sound waves, this transition – called a *shock* – is not a reversible process, and it can transmit information faster than the sound speed. The irreversibility involves some dissipation, which, in the usual case of neutral gases, can be achieved via collisions between particles; in this case, the width of the transition is thus the scale at which the ideal fluid equations no longer hold, that is the mean free path of particles for collisions.

We have defined a shock as a large amplitude irreversible perturbation enabling propagation of information faster than the small amplitude compressible waves. An extreme case arises when no such waves do exist. Figure 2.14 illustrates such an example, that I have borrowed from [2], where an insightful introduction to shocks in space may be found. Imagine a stream of equally spaced vehicles on a straight freeway. Now assume that the drivers have fallen asleep, with the speeds of the cars somehow blocked at their original speed $V_1$, whereas a large truck suddenly stops in the middle of the lane. The system evolves as

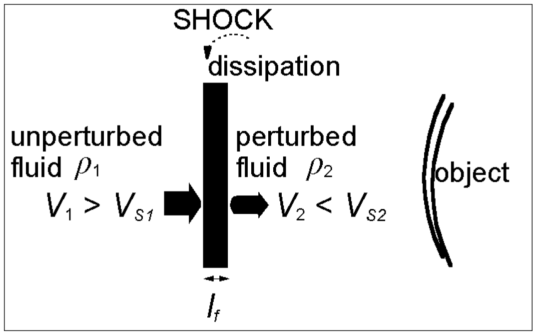


Figure 2.15 A plane shock before a moving object, in the reference frame where the shock (and the object) is at rest. The unperturbed fluid upstream (left) of uniform mass density $\rho_1$ moves at the supersonic speed $V_1 > V_{S1}$ (the upstream sound speed). At the shock, these properties change abruptly to $\rho_2$, $V_2 < V_{S2}$. Between the shock and the object, the fluid slows down smoothly, stopping at the object, and being diverted sideways (not shown). The width of the transition is about the scale $l_f$ at which the ideal fluid equations no longer hold.

shown on the successive rows, from top to bottom. The accumulation of crashes quickly produces two states separated by an abrupt transition: a shock. Upstream, the medium (the regular row of cars) is unperturbed, being unaware of the obstacle ahead. Downstream accumulates a hump of crashed cars. The transition – a shock – moves to the left as more vehicles stop and crash.

This example is a limiting case when there is no small amplitude (reversible) wave propagating information, but it has several basic properties of shocks. The unperturbed medium is supersonic in the sense that it moves faster than the velocity at which information propagates (which is zero here), and its motion becomes abruptly subsonic (zero here) at the shock, where it undergoes an irreversible dissipative transition. The shock is the only way that information can propagate, and it does so at a speed that is effectively supersonic (in the above sense), and is determined by the initial speed of the vehicles, the separation between them and their compression upon crashing.

A more general case is sketched in Fig. 2.15, which is drawn in the frame where the shock (and the object ahead of which it lies) is at rest. Upstream (left), the unperturbed medium has the supersonic speed $V_1 > V_{S1}$ (the sound speed). At the shock the medium undergoes an abrupt dissipative transition, becoming subsonic, of speed $V_2 < V_{S2}$ (the sound speed just downstream of the shock). On this downstream side, the subsonic velocity enables the information on the presence of the obstacle to propagate, so that the speed can decrease smoothly up to the surface of the object where it vanishes. Furthermore, the stream lines are diverted sideways; this is not shown in Fig. 2.15, which is one-dimensional.[18]

[18]Namely, the radius of curvature of the shock (and of the object) is assumed to be much larger than the scales shown.

The fluid properties on both sides of the shock are related by the three ideal fluid equations. First, mass conservation tells us that the mass flux $\rho V$ is the same on both sides of the transition. Second, the equation of motion tells us that the variation in the flux of momentum $\rho V^2$ between both sides balances the variation in pressure $P$. Finally, (adiabatic) energy conservation tells us that the sum of the density of kinetic energy $V^2/2$, plus the enthalpy $5P/(2\rho)$ (with $\gamma = 5/3$) is the same on both sides. This yields

$$\rho_1 V_1 = \rho_2 V_2 \tag{2.79}$$

$$\rho_1 V_1^2 + P_1 = \rho_2 V_2^2 + P_2 \tag{2.80}$$

$$\frac{V_1^2}{2} + \frac{5P_1}{2\rho_1} = \frac{V_2^2}{2} + \frac{5P_2}{2\rho_2} \tag{2.81}$$

where the subscripts 1 and 2 refer respectively to the values upstream and just downstream of the shock. These are called the *Rankine–Hugoniot relations*. The decrease in speed from upstream to downstream is accompanied by an increase in density, pressure and temperature. The Mach number $M = V/V_S$, with the sound speed $V_S = (\gamma P/\rho)^{1/2} = (\gamma k_B T/m)^{1/2}$, changes at the shock from $M_1 > 1$ to $M_2 < 1$, with from (2.79)–(2.81):

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{1 + 3/M_1^2}{4}. \tag{2.82}$$

In the particular case when $V_1 \gg V_{S1}$, we have

$$V_2/V_1 = \rho_1/\rho_2 \simeq 1/4 \tag{2.83}$$

$$V_2/V_{S2} \simeq \sqrt{1/5} \tag{2.84}$$

$$k_B T_2 \simeq m V_1^2/5. \tag{2.85}$$

In that case, one sees that most of the upstream kinetic energy is converted into downstream enthalpy, so that the downstream temperature can be very large.

This holds for neutral gases. In space plasmas, some complications arise, requiring tools that we have not yet introduced. We shall consider these complications when studying shocks in the solar wind (in Section 6.3), and its interaction with solar system objects (Section 7.2) and with the interstellar medium (Section 8.2).

**Transport of momentum and heat**

We now go a step further into the theory and examine how the ideal fluid equations are modified when the fluid is neither isothermal nor adiabatic. Or, from a microscopic point of view, when the particle velocity distribution is not Maxwellian.

Being not Maxwellian has two main consequences.

The first one concerns the property of isotropy, and is relatively trivial. In the presence of a magnetic field $\mathbf{B}$, the velocity distribution tends to acquire a cylindrical symmetry around $\mathbf{B}$, so that the pressure is no longer a scalar, even to zero order, being different in the directions parallel and perpendicular to $\mathbf{B}$. This introduces a complication, but does not present basic difficulties if the velocity distribution remains Maxwellian in the parallel and perpendicular directions; it is then called *a bi-Maxwellian*[19] and defined by two temperatures: $T_\parallel$ and $T_\perp$.

In this case, the (isotropic) adiabatic relation $P \propto \rho^\gamma$ may be generalised in the following way. Just as, for individual particle motions, the ratio of the particle perpendicular energy to the magnetic field $mv_\perp^2/2B$ is an adiabatic invariant, so the ratio of the fluid perpendicular temperature to the magnetic field $T_\perp/B =$ constant. Similarly, we know that the adiabatic invariant of the particle bounce motion is the product of the parallel velocity $v_\parallel$ by the length $L$ of the bounce path. Let us apply this invariance to a magnetic tube of length $L$ and section $s$; conservation of mass yields $\rho L s =$ constant, and conservation of magnetic flux yields $Bs =$ constant. Hence $\rho L/B =$ constant, so that the invariance of $v_\parallel L$ is equivalent to $v_\parallel B/\rho =$ constant. Finally, therefore, for an anisotropic Maxwellian distribution, the adiabatic isotropic relation $P \propto \rho^\gamma$ is replaced by the so-called *CGL relations*[20]

$$T_\perp \propto B \tag{2.86}$$

$$T_\parallel \propto (\rho/B)^2 . \tag{2.87}$$

Unfortunately, this simple scheme does not hold when the velocity distribution is not close to an anisotropic Maxwellian. And still worse, the conditions for the distribution to be an anisotropic Maxwellian are difficult to realise: there must be enough collisions to produce Maxwellians in both the parallel and perpendicular directions, but not so many that the parallel and perpendicular temperatures become equal. Furthermore, even though the gyration around the magnetic field comes to the rescue of collisions for providing quasi-equilibrium in the direction $\perp \mathbf{B}$, this is not so in the direction $\parallel \mathbf{B}$. Therefore, the velocity distributions in space plasmas are generally not bi-Maxwellian, except if the free path for collisions is much smaller than the scale of variation, at least in the direction $\parallel \mathbf{B}$.

We now come to the second consequence of not being Maxwellian. In ordinary gases, small perturbations to the Maxwellian are studied by performing

[19] The bi-Maxwellian distribution has the form $f(\mathbf{v}) = Ae^{-mv_\parallel^2/2k_BT_\parallel} \times e^{-mv_\perp^2/2k_BT_\perp}$, with $A = n\,(m/2\pi k_B)^{3/2}\,T_\parallel^{-1/2}T_\perp^{-1}$, in order to ensure the normalisation $n = \int_{-\infty}^{+\infty} dv_\parallel \int_0^\infty 2\pi v_\perp dv_\perp f(\mathbf{v})$; $v_\parallel$ and $v_\perp$ are the velocity components in the directions respectively $\parallel \mathbf{B}$ and $\perp \mathbf{B}$.

[20] When $B$ is constant, the CGL relation (2.87) yields $T_\parallel \propto \rho^2$, so that the parallel pressure $P_\parallel \propto \rho T_\parallel \propto \rho^\gamma$ with $\gamma = 3$. Since for particles having $N$ degrees of freedom, the adiabatic index $\gamma = 1 + 2/N$, (2.87) then corresponds to an adiabatic compression with 1 degree of freedom. This is not surprising since with constant $B$, the magnetic flux tube must keep a constant section, and thus can only be compressed (or expanded) along $\mathbf{B}$.

an expansion in a small parameter: the ratio of the mean free path $l_f$ to the scale $L$ of variation of the medium, which is related to the extent by which the velocity distribution differs from a Maxwellian. Such an expansion yields the ideal fluid equations, plus small corrections representing a viscosity force and a heat flux. The equation of motion and the Bernoulli theorem are then replaced by

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \bigtriangledown)\,\mathbf{V} \;=\; -\frac{\bigtriangledown P}{\rho} - \bigtriangledown \Phi_G + \mathbf{F}_{vis} \qquad (2.88)$$

$$\boldsymbol{\rho}\mathbf{V} \cdot \bigtriangledown \left( \frac{V^2}{2} + \frac{\gamma}{\gamma - 1}\frac{k_B T}{m} + \Phi_G \right) \;=\; -\bigtriangledown \cdot \mathbf{Q}. \qquad (2.89)$$

The viscosity force $\mathbf{F}_{vis}$ tends to reduce the gradients in velocity. It may be written approximately

$$F_{vis} \simeq \nu \bigtriangledown^2 \mathbf{V} + \nu' \bigtriangledown (\bigtriangledown \cdot \mathbf{V}) \qquad (2.90)$$

where $\nu$ is the *kinematic viscosity*, and the second term vanishes if the medium is incompressible. The heat flux tends to reduce the gradients in temperature, and may be written (in this nearly Maxwellian approximation)

$$\mathbf{Q} \;=\; -\kappa \bigtriangledown T \qquad (2.91)$$

where $\kappa$ is the thermal conductivity. These transport terms are produced by the particle agitation and collisions which enable them to share their momentum and energy. The transport terms therefore increase with the gradients, the random speeds and the free path for collisions. We make below a simplified estimate.

The motion of the particles may be viewed as a random walk at the speed $v_{th}$, with individual random steps of length equal to the collision free path $l_f$. The average distance travelled in this way is zero, but the mean square is not, being $\langle d^2 \rangle = p \times l_f^2$ for $p$ random steps. Travelling a distance $L \gg l_f$ therefore requires a number of steps given by $L^2 = p \times l_f^2$ and therefore a time $\tau \sim p \times l_f / v_{th}$, i.e.

$$\tau \sim \frac{L^2}{v_{th} l_f} \qquad l_f \ll L. \qquad (2.92)$$

This enables us to estimate the coefficient of viscosity and the thermal conductivity. Consider first the equation of motion (2.88), and assume that the main contributions come from the time variation and the shear viscosity, so that in order of magnitude $\partial V/\partial t \sim \nu \bigtriangledown^2 \mathbf{V}$. For a velocity varying at the scale $L$, we have $\mid \bigtriangledown^2 \mathbf{V} \mid \sim V/L^2$, so that $\partial V/\partial t \sim \nu V/L^2$. This means that the viscosity can suppress a velocity variation of scale $L$ in a time $\tau \sim L^2/\nu$. Since this is achieved by the diffusion of particles, which diffuse over a distance $L$ in a time given by (2.92), both times are equal, so that

$$\nu \sim v_{th} l_f \qquad l_f \ll L. \qquad (2.93)$$

The importance of viscosity is quantified by the *Reynolds number*, which represents the ratio of the inertial term $\mid (\mathbf{V} \cdot \bigtriangledown)\,\mathbf{V} \mid$ in the fluid equation of motion (2.88) to the viscosity force. If the velocity varies at the scale $L$ , we have $\mid (\mathbf{V} \cdot \bigtriangledown) \mathbf{V} \mid \sim V^2/L$, while the viscosity force (2.90) is roughly $F_{vis} \sim \nu V/L^2$. The ratio is therefore

$$\frac{\text{inertia}}{\text{viscosity}} \sim VL/\nu \equiv R \qquad \text{(Reynolds number)}. \qquad (2.94)$$

With the expression (2.93) of the viscosity, we have $R \sim (V/v_{th}) \times (L/l_f)$, so that if $V > v_{th}$, the Reynolds number must be much greater than unity to ensure $l_f \ll L$ so that the fluid equation of motion (2.88) holds. Therefore, viscous forces are generally much less important than inertial effects.[21] This is still more true in space and astronomy, because of the extremely large scales.

Consider now the energy equation (2.89). This equation holds in the simple case when there is no time variation, so let us consider another simple case: when the fluid velocity is much smaller than the sound speed so that the dynamical terms are negligible (and the medium behaves as nearly incompressible). In that case, the divergence of the heat flux simply balances the variation in the density of kinetic energy per unit time, so that the energy equation becomes

$$n\frac{\partial}{\partial t}\left(\frac{3}{2}k_B T\right) = \kappa \bigtriangledown^2 T \qquad \text{for} \qquad V \ll V_S. \qquad (2.95)$$

With the order of magnitude estimate $\bigtriangledown^2 T \sim T/L^2$ where $L$ is the scale of variation, (2.95) shows that the heat flux makes the temperature diffuse over a distance $L$ in a time $\tau \sim 3nk_B L^2/(2\kappa)$. Since we have seen that particles diffuse over a distance $L$ in a time given by (2.92), the thermal conductivity is

$$\kappa \sim \frac{3}{2} n k_B v_{th} l_f \qquad\qquad l_f \ll L. \qquad (2.96)$$

Despite the simplicity of our approach, the above estimates of the viscosity and of the thermal conductivity turn out to be accurate to a factor of order unity in a collisional medium.

### Transport in plasmas

How do these results apply in space plasmas? Because of the large ion-to-electron mass ratio, whereas ions and electrons generally have similar temperatures, ions have a much greater kinetic momentum $mv_{th}$ than electrons, but a much smaller kinetic speed $v_{th}$, whereas the free paths are similar. Hence:

- ions transport momentum, and determine the viscosity,
- electrons transport heat, and determine the thermal conductivity.

[21] Beware that the viscosity force, however small quantitatively, may have important qualitative consequences, as we shall see in Section 6.4. Furthermore, we must be careful not to rely too heavily on our intuition, which is based on a familiarity with high Reynolds numbers. This point is nicely addressed by Purcell, E. M. 1977, *Am. J. Phys.* **45** 3.

Because the Reynolds number is generally very large in space plasmas, viscosity is in general negligible.

This is not so, however, for the heat flux. Consider the energy equation (2.89), and compare the transport of heat by thermal conduction, $\bigtriangledown \cdot \mathbf{Q}$, to the transport of heat by the fluid bulk motion, $\propto \rho \mathbf{V} \cdot \bigtriangledown k_B T/m$. For variations at the scale $L$, we make the order of magnitude estimate $\bigtriangledown \sim 1/L$ , which yields

$$\frac{\text{conduction}}{\text{advection}} \sim \frac{\kappa T/L^2}{\rho V k_B T/(mL)} \sim \frac{v_{th} l_f}{VL} \tag{2.97}$$

where we have substituted the expression (2.96) of $\kappa$ and $n = \rho/m$. If the same particle species did produce the viscosity and the heat conductivity, this ratio would be roughly the inverse of the Reynolds number, and would thus be very small. However, heat conduction in plasmas is provided by the electrons, so that the speed in (2.97) is that of electrons, i.e. much greater than that of ions. Furthermore, even though space plasmas have often a bulk motion faster than the ions' most probable speed, the bulk motion is generally slower than the most probable speed of electrons. Hence, heat conductivity is often important in plasmas, even when $l_f \ll L$.

Let us estimate the thermal conductivity in a (collisional) plasma. Substituting the numerical values in (2.96), with $v_{th} = v_{the}$ and the mean free path (2.22), we have

$$\kappa \simeq \frac{10^{-10}}{\ln 1/\Gamma} \times T^{5/2} \ \ \text{W m}^{-1}\ \text{K}^{-1} \tag{2.98}$$

in SI units, where $T$ is the electron temperature. In space plasmas, we have $\ln 1/\Gamma \sim 10$–$20$, so that in order of magnitude $\kappa \sim 10^{-11} \times T^{5/2}$ W m$^{-1}$K$^{-1}$ (SI units). For example, the solar corona, with $T \sim 10^6$ K, has a heat conductivity $\kappa \simeq 10^4$ W m$^{-1}$K$^{-1}$, of the same order of magnitude as the heat conductivity of brass [5].

### Beware of fluid equations in space plasmas

These results, however, must be applied with extreme caution in space plasmas, for two reasons. First, the magnetic field has been neglected. The magnetic field does not affect the thermal conductivity along its direction. However, since in general the particle gyroradii are much smaller than the free paths, the conductivity is strongly reduced in the direction $\perp \mathbf{B}$. Second, even in the direction $\parallel \mathbf{B}$, the thermal conductivity (2.96) only holds when the free path is much smaller than the scales of variation. How much smaller? This question is still not fully solved, and we shall return to it in Sections 4.6 (in the context of the solar corona) and 5.5 (in the context of the solar wind). In practice, values of $l_f/L$ as small as about $10^{-3}$ are required. The basic reason is the fast increase of the particle free path with speed. The free path $l_f$ given by (2.22) is the value for particles of speed equal to the most probable speed. Since for a particle of velocity $v$, the free path $\propto v^4$, particles moving, say, three times faster have a

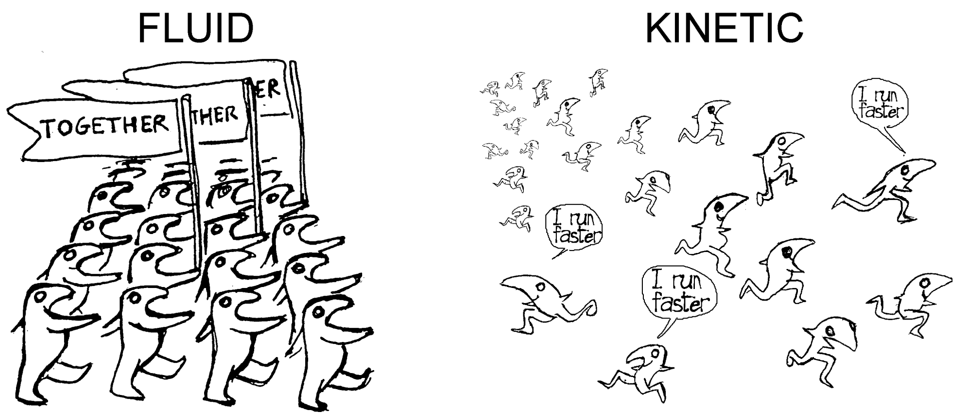


Figure 2.16 The basic difference between the fluid (left) and the kinetic (right) picture, and why the former is often inappropriate in space plasmas. The fluid picture implies a process that enables the particles to transport heat in bulk. However, heat is mainly carried by the particles moving faster than average, which are nearly collisionless and require a kinetic description. (Drawing by F. Meyer.)

free path greater by a factor of $3^4 \sim 10^2$. And since heat is transported by the faster particles, these fast particles must have a small free path for (2.96) to hold.

There lies the difficulty of applying fluid equations in space plasmas. The particles that transport heat are those for which the classical theory of heat transport does not apply! Stated in more precise terms, the fast variation of the free path with speed prevents the expansion of the moment equations in terms of the small parameter $l_f/L$ to converge. Even though the fluid continuity equation and equation of motion do hold (provided the particle pressure is roughly a scalar, and including the electromagnetic contribution if it is not negligible), their solution requires another (local) fluid equation: the energy equation, which generally does not hold because the heat flux is not a simple function of the local derivatives. Another way of understanding this point is to note that for the heat flux to depend on a local derivative, there must be a process – such as collisions – that effectively localises the particles (Fig. 2.16).[22]

It is often said that the fluid picture nevertheless applies because the particles are localised by the gyration around the magnetic field and by the various plasma instabilities. However, the gyration of the particles only localises them in the direction perpendicular to the magnetic field, and the instabilities drive them towards stable configurations, which do not necessarily correspond to local thermal equilibrium.

### 2.3.3 Elements of magnetohydrodynamics

So far, we have ignored the contribution of the electromagnetic force to the fluid motion. This is permissible if the electric charge and currents carried by

[22]Adapted from Meyer-Vernet, N. 2001, *Planet. Space Sci.* **49** 247.

the plasma particles are vanishing or negligible. We have seen that the large-scale electric charge is generally negligible, but this is not necessarily so for the current. And since currents produce magnetic fields, which themselves act on the currents, the particles and the magnetic field are closely coupled. This confers special properties to the medium, which are generally studied with a fluid description. This is the subject of *magnetohydrodynamics* (MHD).

We neglect the charge separation, which is permissible at timescales $T \gg 1/\omega_p$ and spatial scales $L \gg L_D$, and consider the plasma as a single fluid moving at the non-relativistic speed $V \ll c$ and carrying the electric current $\mathbf{J}$. We also assume that the transport coefficients are similar in the directions parallel and perpendicular to $\mathbf{B}$. In the presence of an electric current, a further transport coefficient acts: electric conductivity, which tends to reduce the gradients in electric potential.

### Plasma electric conductivity

The origin of the electric current is the slight difference in bulk motion of ions and electrons in the presence of an electric field $\mathbf{E}$. In the absence of a magnetic field, each electron is accelerated as $m_e d\mathbf{v}/dt = -e\mathbf{E}$. The ions, being more massive, are less easily accelerated, producing a slight difference $\Delta\mathbf{v}$ between the electron and ion velocities, and thus an electric current of density

$$\mathbf{J} = -ne\Delta\mathbf{v} \tag{2.99}$$

for $n$ electrons and $n$ (singly charged) ions per unit volume. Because of electron–ion collisions, of frequency $\nu_{ei}$, the electrons lose their velocity excess $\Delta\mathbf{v}$ in an average time $\Delta t \simeq 1/\nu_{ei}$. At equilibrium (which requires $\mathbf{E}$ weak enough, see Section 5.4.5), the momentum gained by an electron per second $-e\mathbf{E}$ is balanced by the momentum transferred per second to the ions, $m_e\Delta\mathbf{v}/\Delta t$, so that

$$-e\mathbf{E} = m_e\Delta\mathbf{v} \times \nu_{ei}.$$

Eliminating $\Delta\mathbf{v}$ by using (2.99), we deduce $\mathbf{J} = \left(ne^2/m_e\nu_{ei}\right)\mathbf{E}$. This may be written

$$\mathbf{J} = \sigma\mathbf{E} \tag{2.100}$$

$$\sigma = \frac{ne^2}{m_e\nu_{ei}} \qquad \text{(electric conductivity).} \tag{2.101}$$

Substituting $\nu_{ei} = v_{the}/l_f$ with the free path (2.23) and $\ln(1/\Gamma) \sim 10$–$20$, we find $\sigma \sim 3 \times 10^{-4} \times T^{3/2}$. A more exact calculation yields

$$\sigma \simeq 6 \times 10^{-4} \times T^{3/2} \quad (\Omega\ \mathrm{m})^{-1}. \tag{2.102}$$

With a temperature $T \simeq 10^6$ K, this yields $\sigma \simeq 0.6 \times 10^6\ \Omega^{-1}\ \mathrm{m}^{-1}$, nearly equal to the electric conductivity of mercury. Note that $\sigma$ depends only on the temperature, being independent of the density. This is not surprising since with more particles, the current increases, but the collisional losses increase in the same proportion.

An important remark is in order. The above estimate assumes the electrons to follow straight lines between two collisions, and thus neglects the effects of the magnetic field. This is permissible only in the direction $\parallel \mathbf{B}$, or if the mean free path is much smaller than the radius of gyration, so that the curvature of the trajectories may be neglected. Therefore, (2.101) represents the electric conductivity:

- in the direction $\parallel \mathbf{B}$,
- in the directions $\perp \mathbf{B}$, if $l_f \ll r_g$.

In general, the opposite inequality holds, so that the particle gyration reduces strongly the electric conductivity in the directions $\perp \mathbf{B}$.

### Magnetic diffusion

Let us consider an important consequence of the electric conductivity.

For slow time variations and non-relativistic plasma bulk speeds, we may neglect the term $\left(1/c^2\right) \partial \mathbf{E}/\partial t$ compared to $\nabla \times \mathbf{B}$ in Maxwell's equation (2.30),[23] so that we have

$$\nabla \times \mathbf{E} = -\boldsymbol{\partial} \mathbf{B}/\partial t \tag{2.103}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2.104}$$

in a 'laboratory' frame $\mathcal{R}$. In the frame $\mathcal{R}'$ of a plasma moving at velocity $\mathbf{V}$ with respect to $\mathcal{R}$, the electric field is $\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B}$, so that the electric current is $\mathbf{J} = \sigma \mathbf{E}'$, i.e.

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right). \tag{2.105}$$

This yields $\mathbf{E} = \mathbf{J}/\sigma - \mathbf{V} \times \mathbf{B}$, which we substitute into (2.104), to yield

$$\boldsymbol{\partial} \mathbf{B}/\partial t = - \nabla \times (\mathbf{J}/\sigma - \mathbf{V} \times \mathbf{B}). \tag{2.106}$$

Eliminating $\mathbf{J}$ with the help of (2.104) and using the vector identity $\nabla \times (\nabla \times \mathbf{B}) = - \nabla^2 \mathbf{B}$ (since $\nabla \cdot \mathbf{B} = 0$), we deduce

$$\frac{\boldsymbol{\partial} \mathbf{B}}{\partial t} = \frac{\nabla^2 \mathbf{B}}{\mu_0 \sigma} + \nabla \times (\mathbf{V} \times \mathbf{B}). \tag{2.107}$$

This equation contains two contributions to the magnetic field variation:

- a diffusion, produced by the conductive losses,
- a convection, produced by the plasma bulk motion.

[23] This requires that the timescale $\tau$, length scale $L$ and mean velocity $V$ satisfy $E/\tau c^2 \ll B/L$, i.e. with $E \sim VB$, $\tau \gg LV/c^2$.

For magnetic variations at the scale $L$, we have $\mid \nabla^2 \mathbf{B} \mid \sim B/L^2$, so that the first term on the right-hand side of (2.107) is of order of magnitude $B/\left(\mu_0 \sigma L^2\right)$. Its contribution yields $\partial B/\partial t \sim B/\left(\mu_0 \sigma L^2\right)$, making the magnetic field vary on a timescale

$$\tau_\sigma \sim \mu_0 \sigma L^2 \qquad (2.108)$$

proportional to the square of the spatial scale of variation, as in usual diffusion processes. The second term on the right-hand side of (2.107) is produced by the bulk speed, and is of order of magnitude $VB/L$. Either of these two effects can be dominant, depending on the relevant time and length scales. The ratio between both terms is the non-dimensional number

$$\frac{\text{convection}}{\text{diffusion}} \sim \mu_0 \sigma L V \equiv R_m \qquad \text{(magnetic Reynolds number)}. \qquad (2.109)$$

Magnetic diffusion is therefore negligible if $R_m \gg 1$; $R_m$ is called the *magnetic Reynolds number*, in analogy with the fluid Reynolds number whose value quantifies the importance of viscosity.

Basically, the magnetic field diffuses in a conductive medium because the electric currents produce a joule energy loss, which converts magnetic energy into heat. To understand this, consider the following order-of-magnitude estimate. The rate of energy dissipation per unit volume is $\mathbf{J} \cdot \mathbf{E} = J^2/\sigma$. From Maxwell's equation (2.104), the current corresponding to a magnetic field of scale $L$ is of the order of magnitude: $J \sim B/\mu_0 L$, and dissipates energy at the rate $\left(B/\mu_0 L\right)^2/\sigma$ per unit volume. During the diffusion time $\tau_\sigma \sim \mu_0 \sigma L^2$, the energy dissipated per unit volume is thus $\sim B^2/\mu_0$, equal (in order of magnitude) to the initial density of magnetic energy.

There is a major difference between laboratory experiments – on which our intuition is based – and astrophysics. In the laboratory, we have $R_m < 1$ so that magnetic diffusion dominates, and diffusion acts so quickly that the electric currents are mainly determined by the electric conductivity. In astrophysics, the opposite inequality holds because of the large scales and velocities.

### Frozen-in magnetic field

When the magnetic Reynolds number is so large that the electric conductivity may be considered as infinite, the induction equation (2.107) reduces to[24]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}). \qquad (2.110)$$

Consider a closed contour drawn in the fluid, and the magnetic flux that it embraces. As the fluid moves, the contour is displaced and deformed, but one may prove from (2.110) that the flux embraced remains constant. This is known as *Alfvén's theorem*, and is picturesquely expressed by saying that the magnetic

[24] The vorticity field $\nabla \times \mathbf{V}$ satisfies the same equation as $\mathbf{B}$ in the limit of an infinitely large Reynolds number (no viscosity).

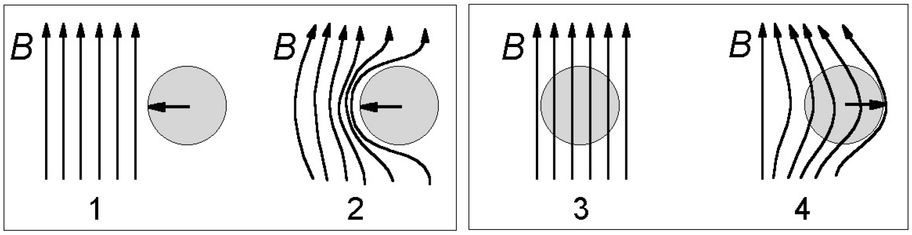


Figure 2.17 A consequence of flux freezing. If we try to introduce a piece of conductor into a magnetic field, the magnetic field lines bend away, avoiding the conductor (1, 2), until magnetic diffusion lets the magnetic field penetrate (3). Conversely, if we try to remove the bar from the magnetic field, the field lines follow the motion, remaining frozen into the conductor (4).

lines of force are 'frozen' in the fluid. The fluid can move freely along the magnetic field lines, but any motion of the fluid perpendicular to the field lines carries them with the fluid.[25]

Basically, this is because if the fluid moves across the magnetic field **B**, the motion induces an electric field of amplitude proportional to the component of the velocity $\perp \mathbf{B}$. If the conductivity is infinite, the electric field must vanish for the electric current to remain finite, and so does this velocity component.

An important consequence of magnetic flux freezing is that in a conducting fluid, one may increase the magnetic field by stretching the field lines. Indeed, consider a small magnetic flux tube of section $s$ and length $l$, that is carried and deformed as the fluid moves. Alfvén's theorem tells us that the magnetic flux across the tube, which is constant along the tube at each time, remains constant too as the tube moves with the fluid, so that $Bs =$ constant. Conservation of mass yields $\rho sl =$ constant, so that $B \propto \rho l$. If the velocity is much smaller than the sound speed, the density $\rho$ remains roughly constant, so that $B \propto l$, i.e. the magnetic field strength increases with the length of the tube. We shall see in Section 3.3 that this property has important applications in the production of the cosmic magnetic fields.

The freezing of the magnetic field in a conductor is a concept that has important consequences in astrophysics, where we have generally $R_m \gg 1$, but to which we are not accustomed in the laboratory where the opposite inequality generally holds. However, similar effects hold to a certain extent, albeit on different scales, with the conductors we encounter in ordinary life (Fig. 2.17). If we try to insert a copper bar, 0.1 m thick, say, in a magnetic field, the magnetic field lines do not penetrate it immediately, because the electric currents induced in the conductor produce a magnetic field opposing the external magnetic field.

[25]Saying that field lines are moving with the fluid is a way of identifying the motion of field lines rather than a statement of fact, since this motion cannot be defined unambiguously from electromagnetic theory alone; the concept of moving magnetic field lines might produce apparent paradoxes if it is applied without care.

It takes about 1 s for the currents to die away so that the external field enters the bar. Meanwhile, the field lines are deformed as shown (Fig. 2.17, left, and Problem 2.5.5). If you quickly pull the bar out of the magnetic field, electric currents are again induced, tending to trap the magnetic field within the bar. The magnetic field lines move with the bar, remaining inside for about 1 s, until the decay of the currents enables the magnetic field to disappear from the bar (Fig. 2.17, right).

This manifestation of magnetic field freezing in conductors occurs on much larger scales in astrophysics, so that the time of field decay is much larger than the timescale of motion. An important consequence is that plasmas in space, remaining tied to the magnetic field lines, do not mix easily across the magnetic field.

In the particular case when the magnetic field does not vary with time, Maxwell's equation (2.29) yields $\bigtriangledown \times \mathbf{E} = 0$, whence $\mathbf{E} = -\bigtriangledown \Phi_E$ where $\Phi_E$ is the electric potential, so that $\mathbf{E}$ is perpendicular to equipotential surfaces. With the approximation $\mathbf{E} \simeq -\mathbf{V} \times \mathbf{B}$, $\mathbf{E}$ is perpendicular to both $\mathbf{V}$ and $\mathbf{B}$, so that $\mathbf{V}$ and $\mathbf{B}$ lie on equipotential surfaces. Hence in this case, stream lines and magnetic field lines are equipotential.

### Magnetic forces

Consider now the electromagnetic force that must be added in the fluid equation of motion (2.63), when the plasma electric currents are not negligible.

With a vanishing large-scale electric charge, the electromagnetic force per unit volume is $\mathbf{J} \times \mathbf{B}$. Eliminating the current density with the aid of Maxwell's equation (2.104), the force per unit volume is

$$\frac{1}{\mu_0} (\bigtriangledown \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0} (\mathbf{B} \cdot \bigtriangledown) \mathbf{B} - \bigtriangledown \left( \frac{B^2}{2\mu_0} \right). \tag{2.111}$$

This is the superposition of:

- a tension force along the field lines equal to $B^2/\mu_0$ (per unit cross-section area normal to them),
- the gradient of a magnetic pressure equal to $B^2/2\mu_0$.

The magnetic force acting on a conducting medium may therefore be pictured in two equivalent ways. The force is the sum of the Lorentz forces $q\mathbf{v} \times \mathbf{B}$ acting on all the moving charges. Alternatively, since the charges are equivalent to a current, itself related to the magnetic field, the force can be described in terms of stresses in the magnetic field. Let us examine these stresses in more detail.

The magnetic tension arises from the curvature of the field lines, and vanishes when they are straight since in that case $(\mathbf{B} \cdot \bigtriangledown) \mathbf{B} = 0$. To understand its origin, consider the simple case of a magnetic field having a cylindrical symmetry, as the one produced by a current flowing along an axis. In this case, the magnetic field follows circles perpendicular to the axis, and depends only on the distance $r$ from

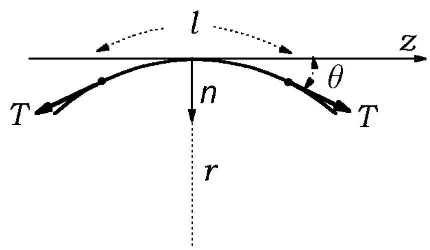


Figure 2.18 The magnetic tension along the field lines may be pictured as a tension force acting on an elastic wire.

this axis. The contribution of the tension to the volume force is $(\mathbf{B}\cdot\bigtriangledown)\,\mathbf{B}/\mu_0 = \mathbf{n}B^2/\mu_0 r$, where $\mathbf{n}$ is a unit vector pointing towards the axis (Fig. 2.18). This contribution is produced by the magnetic tension acting along the field lines, just as for a stretched string. Indeed, when a piece of string of small length $l$ and cross-section area $s$ is stretched with a tension force $\mathbf{T}s$, producing a radius of curvature $r$ (Fig. 2.18), the (downward) vertical force at each extremity is $Ts \times \sin\theta \simeq Ts\theta$, whereas the net horizontal force vanishes. The net force (normal to the string) is $2Ts\theta \simeq Tsl/r$, since the length $l \simeq 2r\theta$, so that the force per unit volume is $T/r$. The magnetic tension $T = B^2/\mu_0$ can therefore be viewed as the tension force per unit cross-section normal to the field lines.

In the general case when the magnetic field strength varies along the field lines, the term $(\mathbf{B}\cdot\bigtriangledown)\,\mathbf{B}$ has also a component $\parallel \mathbf{B}$, which balances the component $\parallel \mathbf{B}$ of the gradient in magnetic pressure, so that the net magnetic force is $\perp \mathbf{B}$ (as it should be), and may be expressed as

$$\frac{B^2}{\mu_0}\frac{\mathbf{n}}{R_c} - \bigtriangledown_\perp\left(\frac{B^2}{2\mu_0}\right) \qquad \text{(magnetic force per unit volume)} \qquad (2.112)$$

where $\mathbf{n}$ is a unit vector pointing towards the centre of curvature, $R_c$ is the radius of curvature and $\bigtriangledown_\perp$ denotes the component of the gradient in the plane $\perp \mathbf{B}$. The magnetic tension tends to oppose the curvature of the field lines and to shorten them, just as does the tension of an elastic string. The magnetic pressure tends to oppose the compression of the field lines and to expel the plasma from regions of high magnetic field, just as the ordinary gas pressure tends to expel matter from high pressure regions. Hence, the plasma and the magnetic field conspire to keep the plasma+magnetic pressure constant, by putting matter where the magnetic field is weak and vice versa.

Finally, therefore, the equilibrium and dynamics of a magnetised plasma are governed by three terms:

- inertia, corresponding to the density of bulk kinetic energy $\rho V^2/2$,
- thermal pressure, corresponding to the density of random kinetic energy $\sim \rho k_B T/m$,
- magnetic forces, corresponding to the density of magnetic energy $B^2/2\mu_0$.

If the first term dominates the third, the motion is not significantly affected by the magnetic field, and it controls the field lines. In contrast, if the third term dominates, the magnetic field controls the motion. Equating the first and third terms gives a speed

$$V = \frac{B}{(\mu_0 \rho)^{1/2}} \equiv V_{\mathrm{A}} \qquad \text{(Alfvén speed)}. \tag{2.113}$$

This is the typical speed to which the magnetic field can accelerate the plasma; we shall return to it later.

On the other hand, when the bulk velocity of the medium vanishes, the nature of the equilibrium is determined by the ratio of the second to the third terms:

$$\beta = \frac{nk_B T}{B^2/2\mu_0}. \tag{2.114}$$

In that case, the equilibrium is governed by the magnetic field if $\beta \ll 1$, and by the plasma if $\beta \gg 1$.

### Magnetohydrodynamic waves

Just as a stretched string supports waves, in which transverse motions produced by the string's tension propagate along the string, so a magnetised plasma supports transverse waves, known as *Alfvén waves*, in which transverse motions of the field lines produced by the magnetic tension propagate along the field lines.

On a stretched string, the phase speed is $v = \sqrt{T/\rho}$, where $T$ is the tension force per unit cross-section area, and $\rho$ is the mass density of the string. Similarly, the phase speed of an Alfvén wave is $\sqrt{T/\rho}$, where the tension $T = B^2/\mu_0$ is produced by the magnetic field, and the mass density $\rho$ is provided by the plasma which moves with the field lines because of flux freezing. With this analogy, the phase speed is $\sqrt{B^2/\mu_0\rho}$, the Alfvén speed defined in (2.113), and oriented along the field lines. This result may also be found as follows.

Consider a magnetic field oriented along $\mathbf{z}$, i.e. $B = B_z$, and deform the field lines as shown in Fig. 2.18. If $x(z)$ is the amplitude of the displacement normal to $\mathbf{z}$, the magnetic field component along $\mathbf{x}$ is

$$B_x = B_z dx/dz \simeq B dx/dz. \tag{2.115}$$

The magnetic tension produces a restoring force per unit volume given by the first term of (2.112) as $F_x = B^2/(\mu_0 R_c)$ where the radius of curvature $R_c = \left(d^2x/dz^2\right)^{-1}$. Since the field lines move together with the plasma of mass density $\rho$, this yields the equation of motion

$$\rho \frac{d^2x}{dt^2} = \frac{B^2}{\mu_0}\frac{d^2x}{dz^2}. \tag{2.116}$$

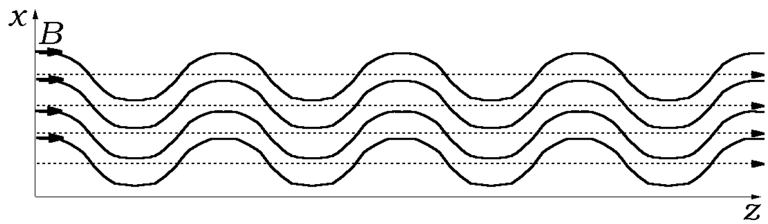


Figure 2.19 Sketch of the magnetic field lines in an Alfvén wave; the undisturbed magnetic field lines are shown as dotted lines.

There are two solutions of the form[26]

$$x \propto e^{-i(\omega t - kz)} \qquad \omega/k = \pm V_A \qquad V_A \equiv B/\sqrt{\mu_0 \rho} \tag{2.117}$$

which are plane waves propagating along **z** at the phase speed $V_A$, perturbations of **B** that are $\perp\mathbf{B}$.

The material moves with the field line at right angles to the direction of propagation at the speed $v = dx/dt = -i\omega x$ (in Fourier space), so that the kinetic energy per unit volume is $\rho \mid v^2 \mid /2 = \rho\omega^2 \mid x^2 \mid /2$. From (2.115) and (2.117), the magnetic field variation in the wave is $B_x = Bikx$, so that the wave magnetic energy per unit volume is $\mid B_x^2 \mid /2\mu_0 = k^2 B^2 \mid x^2 \mid /2\mu_0$. Since $\omega/k = V_A$, both energies are equal.

The flux of energy carried by the wave is thus equally shared by the mechanic and magnetic energies, and equal to twice the flux of mechanical energy $V_A \times \rho \mid v^2 \mid$.[27]

Since the wave vector **k** is normal to the fluid velocity **v**, we have in Fourier space $\bigtriangledown \cdot \mathbf{v} = i\mathbf{k} \cdot \mathbf{v} = 0$, so that the continuity equation yields $\rho =$ constant. Hence, the mass density is not perturbed in these waves, as might be expected from Fig. 2.19, which shows that the deformation of the field lines does not change the volume of the flux tubes.

Finally, even though our calculation assumes small perturbations, these waves can exist with a large amplitude, provided the medium is incompressible and adiabatic. In this case, Alfvén waves may propagate at constant speed in a homogeneous medium without any distortion or attenuation. These waves are of great importance in astronomy, as they transport perturbations along the magnetic field over long distances.

The above calculation assumes a frozen-in magnetic field and non-relativistic speeds, which require in particular $k \ll 1/r_g$, $\omega \ll \omega_g$ and $V_A \ll c$, and consider a special geometry: perturbations of speed and magnetic field that are $\perp$ **B** and propagate along **B**. For other directions, one finds three MHD modes: a generalisation of the Alfvén wave propagating at an angle with the magnetic

[26] In Fourier space, with the usual convention that the physical quantity corresponding to, say, the displacement noted $x$ in Fourier space is equal to the real part of $x$.

[27] We have neglected the terms involving the electric field in Maxwell's equations, so that the electric energy is absent. This approximation is acceptable if $V_A \ll c$.

field, and two modes which, contrary to the Alfvén mode, involve some plasma compression, so that the restoring forces are the magnetic stresses plus the gas pressure gradient.

The generalised Alfvén wave has a wavefront that is not necessarily $\perp$ **B**, and propagates at the Alfvén speed $V_A$ in the direction $\parallel$ **B** whatever the angle $\theta$ between **k** and **B**, so that $\omega/k = V_A \cos\theta$; the perturbations of velocity and magnetic field are perpendicular to both **k** and **B**, so that a given magnetic field line still looks like a plucked string.

Of the two compressive modes, one propagates faster than the other, so that they are called the fast and slow *magnetosonic waves*. In the so-called *fast wave*, the particle pressure and the magnetic forces act roughly in phase, and the propagation depends strongly on the angle $\theta$ between **k** and **B**:

- When $\theta \to 0$ the phase speed $\omega/k \to \max(V_A, V_S)$.
- When $\theta \to \pi/2$ the velocity perturbation is $\parallel$ **k** just like a sound wave, with the effect of the gas pressure supplemented with that of the magnetic pressure. This is thus a longitudinal wave propagating $\perp$ **B**, in which the field lines move parallel to themselves, with alternating compressions and rarefactions of the gas and field, so that the restoring force is produced by the sum of the gas and the magnetic pressure. The phase speed may be calculated by generalising our calculation of the sound waves (2.76), as $(\gamma P/\rho + \gamma_M P_M/\rho)^{1/2}$ where $P_M = B^2/2\mu_0$ is the magnetic pressure and $\gamma_M$ is the corresponding adiabatic index ($\gamma_M = 2$ for this two-dimensional compression normal to **B**). Therefore, $\omega/k \to \left(V_A^2 + V_S^2\right)^{1/2}$.

The so-called *slow wave* has the restoring forces acting roughly out of phase:

- When $\theta \to 0$ the phase speed $\omega/k \to \min(V_A, V_S)$.
- When $\theta \to \pi/2$ the phase speed vanishes.

### Non-ideal magnetohydrodynamics

The concept of a frozen-in magnetic field has been derived by assuming that the electric field vanishes in the plasma frame. This is not exactly so, for several reasons.

First of all, even if the conductivity is very large, it is never infinite. This point is not a mere semantic distinction because the electric resistivity, however small, makes the magnetic field diffuse on a timescale that is proportional to the square of the length scale. Diffusion thus acts very quickly if small scales arise, even with a very large conductivity. Let $L$ be the typical scale of variation, so that $R_m \gg 1$ and the diffusion term $\nabla^2\mathbf{B}/\mu_0\sigma$ in (2.107) is negligible. Now suppose that some effect produces a variation at a smaller spatial scale $l \ll L$ so that $\mu_0\sigma l V < 1$. In that case, the diffusion term $\nabla^2\mathbf{B}/\mu_0\sigma \sim B/\left(\mu_0\sigma l^2\right)$

becomes larger than the motional term $\nabla\times(\mathbf{V}\times\mathbf{B})\sim VB/l$, so that the electric conductivity is no longer negligible.[28]

With a finite electric conductivity, there is an electric field in the frame of the plasma equal to $\mathbf{J}/\sigma$. This is not, however, the only component of the electric field, because in deriving (2.100)–(2.101), we have neglected:

- the gradient of the electron pressure $P_e$, which produces a contribution to the electric field in the plasma frame equal to $-\nabla P_e/ne$, which balances the electron pressure force, to maintain the plasma neutral,
- the effect of the magnetic field on the electron trajectories, which produces an additional contribution to the electric field equal to $\mathbf{J}\times\mathbf{B}/ne$, in order to produce a force on the electrons that balances the Lorentz force,[29]
- the effects of the electron inertia on the current, which produces an additional contribution to the electric field of order of magnitude $VBc/(\omega_p L)$, which is therefore negligible for scales $L > c/\omega_p$.[30]

We shall see that the electron pressure, and the corresponding large-scale electric field, play an important role in the solar wind.

Another important consequence of the finite electric field is that the breakdown of the frozen magnetic field concept may produce important changes in the topology of the magnetic field. This is called magnetic *reconnection*, and is sketched in Fig. 2.20.[31] When field lines of different directions are pushed together, producing large gradients in a magnetic field, the magnetic field may disappear quickly in a small region, the magnetic energy being converted into other forms of energy, and the lines reconnect to form a new topology, so that the connectivity of plasma parcels by field lines changes. This enables the magnetic field to pass to a state of lower energy, releasing energy in producing plasma jets and high-energy particles, in addition to heating. This change in topology of the field lines enables different plasmas to mix.

Normal magnetic dissipation acts at the timescale $\tau_\sigma\sim\mu_0\sigma L^2$. This may be compared to the collision time, which may be written, with the aid of (2.101), as

$$\frac{1}{\nu}\sim\tau_\sigma\times\left(\frac{c/\omega_p}{L}\right)^2$$

[28] A similar effect acts in hydrodynamics with the viscosity. In a non-viscous fluid, objects move without friction. Nevertheless, the friction force on an object of cross-section $S$ moving at speed $V$ in a fluid of density $\rho$ is of order of magnitude $\rho V^2 S$ for a very large range of fluid viscosities including extremely small ones. We will return to this apparent paradox in Section 6.4.

[29] This term is called the *Hall electric field*. When it is not negligible, but the other terms are, the electric field in the laboratory frame is $\mathbf{E}=-(\mathbf{V}-\mathbf{J}/ne)\times\mathbf{B}$. If $\mathbf{V}_i$ and $\mathbf{V}_e$ are respectively the bulk speeds of ions and electrons, the current is $\mathbf{J}=ne(\mathbf{V}_i-\mathbf{V}_e)$, so that since $\mathbf{V}\simeq\mathbf{V}_i$ because the ions carry the mass, $\mathbf{E}=-\mathbf{V}_e\times\mathbf{B}$ and the frozen-in approximation holds for the electron gas, instead of holding for the plasma as a whole.

[30] The so-called *electron inertial length*.

[31] Beware that Fig. 2.20 is a simplistic view that not only ignores the three-dimensional nature of the phenomenon, but uses an MHD concept (moving field lines to which the plasma is attached) under conditions when it does not apply.

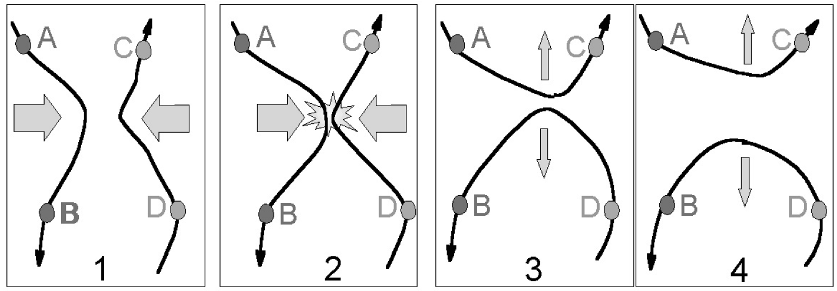


Figure 2.20 A naive view of magnetic reconnection. Two field lines of opposite directions are pushed together; the fluid parcels A and B lie on one field line, C and D lie on the other one (1). After reconnection (2), the fluid parcels A and C lie on a field line, the fluid parcels B and D lie on another one (3), which are separating from each other (4).

so that for scales greater than the electron inertial length $c/\omega_p$ (at which the electron inertia may be neglected), the collision time is smaller than the time $\tau_\sigma$ for magnetic resistive dissipation.

In nearly collisionless plasmas, small scales arise, which may be smaller than all the plasma characteristic scales: the gyroradii and inertial lengths for all particle species, and even the Debye length. In this case, all the fluid and MHD approximations break down, and one must describe the plasma in a kinetic way. Reconnection then acts faster than the scales $\tau_\sigma$ and $1/\nu$, acting instead on timescales that are typically a fraction of the *Alfvén time* (the time of displacement at the Alfvén speed)

$$\tau_A = L/V_A = (\tau_\sigma/R_M) \times (V/V_A) . \tag{2.118}$$

### 2.3.4 Waves and instabilities

We have seen that perturbations in the magnetic field and/or the plasma pressure may drive several kinds of waves. In deriving the properties of these waves, however, we have neglected the electron inertia, and we have pictured the plasma as a fluid. Even if the unperturbed medium has Maxwellian velocity distributions, these approximations are not acceptable at frequencies equal to or greater than the plasma characteristic frequencies (the gyrofrequencies and the plasma frequency), or wavelengths smaller than the plasma characteristic scales (the Debye length, the gyroradii and inertial lengths).

#### Electromagnetic waves

At frequencies near the plasma frequency or greater, the inertia of the electrons is no longer negligible, but, because the ions have much greater mass, we may neglect their motion. We also neglect the electron random motion (which is acceptable if the wave phase speed is much greater than the electron most probable speed), and the ambient magnetic field (which is acceptable at frequencies much greater than the electron cyclotron frequency), and assume that the wave weakly perturbs the medium. In this case, we may assume all the electrons to acquire the same velocity $\mathbf{v}$ in the wave electromagnetic field, so that they obey the fluid equation of motion

$$m_e \partial \mathbf{v}/\partial t = -e\mathbf{E} \tag{2.119}$$

where $\mathbf{E}$ is the wave electric field; we have also neglected the Lorentz force produced by the wave magnetic field and the term $(\mathbf{v} \cdot \bigtriangledown)\,\mathbf{v}$ since they are of second order in the (small) perturbation. This electron motion produces a current density

$$\mathbf{J} = -ne\mathbf{v} \quad \Rightarrow \quad \frac{\partial \mathbf{J}}{\partial t} = \omega_p^2 \epsilon_0 \mathbf{E} \tag{2.120}$$

where $n$ is the unperturbed electron density, $\omega_p$ the corresponding plasma frequency, and we have substituted (2.119).

Consider a wave satisfying $\bigtriangledown \cdot \mathbf{E} = 0$, so that from Maxwell's equation (2.29), the density of electric charge vanishes. In this case, we have $\bigtriangledown \times (\bigtriangledown \times \mathbf{E}) = -\bigtriangledown^2 \mathbf{E}$, so that Maxwell's equations (2.29) and (2.30) yield

$$\bigtriangledown \times \frac{\partial \mathbf{B}}{\partial t} = \bigtriangledown^2 \mathbf{E} = \mu_0 \frac{\partial \mathbf{J}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Substituting the current (2.120), this yields

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \bigtriangledown^2 \mathbf{E} = -\omega_p^2 \mathbf{E} \tag{2.121}$$

which has a plane wave solution $\propto e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$ with

$$\omega^2 = \omega_p^2 + k^2 c^2. \tag{2.122}$$

From Maxwell's equations (2.29), and since we have assumed $\bigtriangledown \cdot \mathbf{E} = 0$, this is a transverse wave ($\mathbf{k} \perp \mathbf{E} \perp \mathbf{B}$) as is light propagating in vacuum, but we see from (2.122) that the wave has a phase speed $\omega/k = c/\sqrt{1 - \omega_p^2/\omega^2}$ that depends on the frequency and wave number, and propagates only at frequencies greater than the plasma frequency; for $\omega < \omega_p$, $k$ is imaginary, so that the wave decays in space, as does light reflected from a mirror. Since the phase speed $\omega/k > c$, it is generally much greater than the speeds of individual particles, which can thus be safely neglected. This is why the wave is neither affected by the pressure of the particles nor damped.[32]

If the ambient magnetic field is not negligible, the electrons gyrate in this field. This affects the wave for frequencies of the order of magnitude of the electron gyrofrequency or smaller. In particular, for propagation along the ambient

[32] In the absence of collisions.

magnetic field, the wave is split into two waves in which the electrons and the wave electric (and magnetic) field gyrate around the ambient magnetic field at the wave frequency.[33] The mode rotating in the same sense as do the electrons in the ambient magnetic field (the direct sense) is more easily emitted and absorbed by them. In the frequency range $\omega < \omega_{ge} < \omega_p$, one finds that the delay of propagation increases towards low frequencies, giving rise to a characteristic whistle, so that this mode is called a *whistler*.

### Langmuir waves

The electromagnetic waves studied above have $\nabla \cdot \mathbf{E} = 0$, so that there is no variation in the density of electric charge. We have seen in Section 2.1 that variations in the density of electric charge make electrons oscillate in bulk when the random agitation is negligible. The random agitation has two major consequences. The first consequence is that it produces a pressure force that makes these oscillations propagate. This may be understood by picturing the electrons as a fluid of number density $n$ and pressure $P = nk_BT$, so that their equation of motion in the wave field is

$$m_e \partial \mathbf{v}/\partial t = -e\mathbf{E} - \nabla P/n \tag{2.123}$$

in the absence of ambient magnetic field. Because of the small timescale, we assume the electrons to behave as an adiabatic fluid (with 1 degree of freedom – the wave direction of propagation), so that $P \propto n^\gamma$ with $\gamma = 3$. This yields $\nabla P = 3k_BT \nabla n$. Because of the pressure term, we now have instead of (2.120)

$$\frac{\partial \mathbf{J}}{\partial t} = \omega_p^2 \epsilon_0 \mathbf{E} + \frac{3}{2} v_{the}^2 e \nabla n. \tag{2.124}$$

Because of the small timescale, the ions do not move, so that the variation in electric charge density is $\partial \rho_e/\partial t = -e\partial n/\partial t$; hence the continuity equation applied to the electric charge and current yields $\nabla \cdot \mathbf{J} = e\partial n/\partial t$. Taking the divergence of (2.124) and expressing $\mathbf{J}$, $\mathbf{E}$ and $\nabla n$ in terms of the charge density $\rho_e$ with the help of Maxwell's equation $\nabla \cdot \mathbf{E} = \rho_e/\epsilon_0$, we find

$$\frac{\partial^2 \rho_e}{\partial t^2} - \frac{3}{2} v_{the}^2 \nabla^2 \rho_e = -\omega_p^2 \rho_e. \tag{2.125}$$

This equation has a plane wave solution $\propto e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$ with

$$\omega^2 \simeq \omega_p^2 + 3k^2 v_{the}^2/2 \simeq \omega_p^2 \left(1 + 3k^2 L_D^2\right). \tag{2.126}$$

[33]These waves have respectively a *right-hand* and a *left-hand* circular polarisation, with respect to the direction of the ambient magnetic field. Beware that a number of different conventions are in use to label these waves, so that the same wave is given a different handedness depending on the context. For physicists, right-hand and left-hand generally refer to the direction of the wave vector $\mathbf{k}$; for radio astronomers, the same words refer to the direction from which the wave is coming, i.e. $-\mathbf{k}$, so that plasma physicists and radioastronomers use the same label only if $\mathbf{B}$ and $\mathbf{k}$ have opposite directions; furthermore, old textbooks call these waves respectively *ordinary* and *extraordinary*, whereas the latter names now denote electromagnetic waves of linear polarisation that propagate at an angle to the magnetic field.

The wave electric field (and also the particle velocity perturbation due to the wave) is parallel to **k**; it is due to the charge separation as the electrons oscillate whereas the massive ions are barely set in motion at these high frequencies. Whereas the electromagnetic wave found previously is a simple generalisation of the electromagnetic wave in vacuum, with the plasma acting as a dielectric medium (of refractive index $\sqrt{1-\omega_p^2/\omega^2}$ for $\omega \gg \omega_{ge}$ and a birefringent medium otherwise), this longitudinal wave is entirely new. It is called a *Langmuir wave*, and is simply the Langmuir oscillation that propagates because of the electron thermal motion. One sees from (2.126) that the wavelength is greater than $L_D$ for $\omega \sim \omega_p$, and tends to infinity as $\omega \to \omega_p$ where the wave reduces to a plasma oscillation. We shall see an illustration of these properties in Section 6.4.

### Landau damping

The second consequence of the thermal agitation is that the Langmuir waves are damped. This is called *Landau damping*. This damping does not appear in (2.126), which gives a real value of $k$ for any frequency $\omega > \omega_p$, because our derivation pictured the electrons as a fluid, while Landau damping comes from the individual motions of the particles and therefore requires a kinetic picture. This damping process is subtle since it produces losses without introducing any explicit damping term in the equation of motion, and appears in a wide range of problems outside plasma physics, from Saturn's rings to fireflies.[34]

A simple way of understanding this process is to picture it as a resonance between the wave and the electrons whose velocity equals the wave phase speed. These electrons see a constant electric field, and are therefore in resonance with the wave. The electrons moving slightly slower than the wave are accelerated, whereas those moving slightly faster are decelerated. With a Maxwellian velocity distribution (and more generally with a distribution whose derivative is negative for a velocity equal to the phase speed), there are more slower particles than faster ones, so that the net effect is to damp the wave. The damping is greater when there are more electrons having a speed close to that of the wave, which comes true when the phase speed is smaller than or close to the electron thermal speed. One sees from (2.126) that this happens when $k \geq 1/L_D$. Hence, the Langmuir wave propagates at frequencies above but close to the plasma frequency, and is heavily damped at larger frequencies.

Conversely, if the velocity distribution has a positive derivative for a velocity equal to the wave phase speed, the wave grows. This happens for example when a beam of particles of velocity **v** propagates in a plasma faster than the electron thermal speed; the beam excites Langmuir waves of phase speed $\omega/k \simeq v$ directed along **v**, converting the energy of the beam into wave energy. This is an example of one of the numerous instabilities that arise in non-equilibrium plasmas, and we shall see an illustration of it in Section 6.4.

[34] See for example Meyer-Vernet, N. and B. Sicardy 1987, *Icarus* **69** 157, and Sagan, D. 1994, *Am. J. Phys.* **62** 450.

A similar resonance occurs for other kinds of plasma waves when the phase speed coincides with that of plasma particles. An example is cyclotron damping. For particles moving at velocity $v_\parallel$ along the ambient magnetic field $\mathbf{B}$, the wave frequency is Doppler-shifted to the frequency $\omega - k_\parallel v_\parallel$ where $k_\parallel$ is the component of the wave vector along $\mathbf{B}$, so that at some velocity it may coincide with the cyclotron frequency (or a harmonic), i.e. $\omega - k_\parallel v_\parallel = n\omega_g$. The wave is then damped when it has an electric field component perpendicular to $\mathbf{B}$, so that particles experience a perturbing force which oscillates at the cyclotron frequency (or a harmonic).

### 2.3.5 Summary

The fast increase with speed of the collisional free path makes fast particles virtually collisionless, and therefore easily driven out of equilibrium. Two effects come to the rescue of collisions for tending to restore equilibrium: the first effect is the gyration around the magnetic field, but it acts only across the magnetic field; the second one is due to plasma instabilities which, however, only prevent the velocity distributions from becoming too crazy, but do not oblige them to be Maxwellian. This is why dilute plasmas have often velocity distributions which are non-Maxwellian but not too crazy. Since fluid descriptions – including MHD – assume velocity distributions to be nearly Maxwellian (or bi-Maxwellian), dilute plasmas often require a kinetic description.

Both fluid and kinetic descriptions involve the conservation of particles, momentum and energy. Since momentum and energy are carried respectively by plasma ions and electrons, of which the latter move much faster, thermal conductivity generally plays a more important role than viscosity, and is rarely negligible. Therefore, the major difficulty of fluid descriptions is to model correctly the transport of energy. Whereas in kinetic descriptions the heat flux is calculated self-consistently, fluid descriptions use various approximations of it. Basically, three kinds of fluid approximations are made, depending on the conditions: for very slow or very fast changes, the plasma is assumed to be isothermal or adiabatic respectively, corresponding to a heat flux that is respectively infinite or zero; for intermediate cases, the heat flux is approximated by the collisional transport. These approximations are valid only if the particle velocity distributions are close to Maxwellian.

When the electric field in the plasma frame is small enough, the plasma and magnetic field lines may be pictured as being tied together, so that the plasma can only move along the magnetic field but not across it. As a consequence, space plasmas tend to be organised by the magnetic field lines and do not mix easily across them. The magnetic forces on the plasma may be described as the superposition of a pressure acting across the field lines, which tend to expel the plasma from regions of strong magnetic field, plus a tension acting along the field lines, which tends to shorten and unbend them.

## 2.4 Basic tools for ionisation

What is the origin of ionisation in the solar interior, the solar wind and the planetary environments? A full answer to this question requires highly polished calculational techniques using the tools of quantum mechanics, and is outside the scope of this book; we shall give instead order of magnitude estimates based on elementary considerations [14], [20]. We will do so with the naive point of view of merely supplementing classical physics by the Heisenberg uncertainty relations.

### 2.4.1 Energy of ionisation and the size of the hydrogen atom

Let us estimate the size and energy of the hydrogen atom in its most stable state: the fundamental one. The H atom is made of a proton and an electron bound together by an attractive Coulomb force. The potential energy of the electron at distance $r$ from the proton is $W_E = -e^2/(4\pi\epsilon_0 r)$. Because of the small size of the system, the kinetic energy of the electron is determined by Heisenberg's uncertainty relation, which says that an electron confined in a small region of size $\Delta r$ has the momentum $\Delta p \sim \hbar/\Delta r$, whence the kinetic energy $W_{th} = (\Delta p)^2/2m_e$. Since in the H atom the electron is confined in a region of size $\Delta r \sim r$, we have $W_{th} \sim \hbar^2/\left(2m_e r^2\right)$, so that the total energy of the electron is

$$W \sim \frac{-e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{2m_e r^2}. \qquad (2.127)$$

The most stable state is the one of minimum energy, which arises for a distance $r$ so that $dW/dr = 0$, i.e.

$$r \sim \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e} \equiv r_{Bohr} \qquad \text{(Bohr's radius)}. \qquad (2.128)$$

Substituting the numerical constants, we have

$$r_{Bohr} = \hbar/(\alpha m_e c) \simeq 0.53 \times 10^{-10} \text{ m} \qquad (2.129)$$

where

$$\alpha = e^2/(4\pi\epsilon_0\hbar c) \simeq 1/137 \qquad \text{(fine structure constant)}. \qquad (2.130)$$

For $r = r_{Bohr}$, we have $W_{th} = -W_E/2$ in accord with the Virial theorem (Section 3.1.1), and the total energy (2.127) is equal to minus

$$\frac{e^4 m_e}{8\epsilon_0^2 h^2} \equiv W_{Bohr} \qquad \text{(Bohr's energy)}. \qquad (2.131)$$

Substituting the numerical constants, we have $W_{Bohr} \simeq 2.2 \times 10^{-18}$ J, which comes to about 13.6 eV.

This is the energy required to ionise a hydrogen atom from its fundamental state. By mere luck, this order-of-magnitude estimate gives the exact result. For atoms made of a nucleus of charge $Ze$ surrounded by $Z$ electrons, an electron of the outer shell sees the nucleus charge shielded by the charge of the $Z-1$ other electrons, so that the energy required to strip an outer electron is of the same order of magnitude as for ionising hydrogen.

This is no longer true, however, as more electrons are stripped, so that producing highly charged ions requires a large energy. Very crudely, stripping an element of charge $Ze$ of its last electron (of potential energy $-Ze^2/(4\pi\epsilon_0 r)$ at distance $r$), to produce a bare nucleus, requires an energy that is greater than the Bohr energy by the factor $Z^2$.

### 2.4.2 Ionisation by compressing or heating

These results furnish hints as to how a medium may be ionised.

One way is to compress, so that the average distance between ions becomes smaller than the sum of the radii of two atoms; this somehow crashes the atoms. For hydrogen, this happens when the ion number density $n$ satisfies $n^{-1/3} < 2r_{Bohr}$, i.e. when the mass density satisfies $\rho > m_p/(2r_{Bohr})^3 \simeq 1.5\times 10^3$ kg m$^{-3}$. We shall see in the next chapter that the density in the central parts of the Sun largely exceeds this value, producing ionisation.

Less dense media may be ionised by furnishing to atoms the ionisation energy $W_{Bohr}$. This may be done in several ways. One way is to heat. One might think naively that significant ionisation requires heating at a temperature so that the thermal energy $k_B T > E_{Bohr}$. This is, however, a classical point of view, and in fact a smaller energy is required because ionisation increases considerably the phase space accessible to an electron and therefore its number of possible states. This may be understood as follows.

At equilibrium, the ratio of the number of free electrons in some volume of phase space to the number of electrons bound in an H atom is proportional to $e^{-\Delta W/k_B T}$ (where $\Delta W$ is the difference in total energy between both states) times the ratio of the number of possible states for respectively a free electron and a bound one. Let $n_i = n_e$ be the ion (or electron) number density and $n_n$ the number density of neutrals. In the volume $V$, a recombining electron may do so with either of the $n_i V$ ions and have one of two spin states, so that its number of possible states is $2n_i V$. On the other hand, the number of possible states of a free electron at temperature $T$ and thermal speed $v_e \sim (k_B T/m_e)^{1/2}$ is roughly twice the ratio of the available volume $V$ to the 'private' volume of a free electron (that is roughly the cube of its wavelength $h/m_e v_e$). With the approximation $\Delta W \sim W_{Bohr}$, this yields the degree of ionisation at thermal equilibrium $n_e/n_n \sim n_i^{-1}(h/m_e v_e)^{-3} e^{-W_{Bohr}/k_B T}$. A more exact calculation[35] yields nearly the same result:

$$\frac{n_i n_e}{n_n} \simeq \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-W_{Bohr}/k_B T} \qquad (2.132)$$

[35] Integrating over the electron velocity distribution.

which is a simplified version of the so-called *Saha formula*. The term before the exponential is of the order of magnitude of $(T/T_F)^{3/2}$ where $T_F$ is the Fermi temperature (see Section 2.1), which is much smaller than $T$ in non-degenerate media. Hence, in general, ionisation is already significant at a temperature much smaller than $W_{Bohr}/k_B \simeq 1.6 \times 10^5$ K.[36]

Let us apply (2.132) to the solar corona. With $n_e \sim 10^{14}$ m$^{-3}$ and $T \sim 10^6$ K, we find $n_e/n_n \sim 2 \times 10^{16}$, which shows that the corona is virtually completely ionised. One must be careful, however, in applying this formula since the corona is not in thermal equilibrium.

Planetary atmospheres are in general too cold for being thermally ionised.

### 2.4.3 Radiative ionisation and recombination

#### Ionisation and recombination

Another way of ionising particles is to subject them to photons of energy greater than the energy of ionisation.

The ionisation rate per atom is proportional to the flux of ionising photons $F$, and may be written

$$\Lambda_{ion} = F\sigma_{ion} \tag{2.133}$$

in s$^{-1}$. Since the flux of photons $F$ is expressed in m$^{-2}$ s$^{-1}$, $\sigma_{ion}$ – the *cross-section for ionisation* – has the dimension of an area. With a concentration $n_n$ of neutrals, the ionisation rate per unit volume is therefore $dn_i/dt = n_n\Lambda_{ion} = n_nF\sigma_{ion}$.

Once ionised, the ions may recombine with electrons. For hydrogen, the radiative ionisation and recombination processes may be written

$$\mathrm{H} + \mathrm{ph} \rightleftharpoons \mathrm{H}^+ + e^-.$$

For a given ion, electrons recombine at a rate proportional to their flux $\sim n_e v_{the}$ so that the rate of recombination per ion is

$$\Lambda_{rec} = n_e v_{the}\sigma_{rec} = n_e\beta \tag{2.134}$$

in s$^{-1}$, where $\sigma_{rec}$ has the dimension of an area, and may be expressed through the parameter $\beta = v_{the}\sigma_{rec}$, the *recombination coefficient*. At equilibrium, in the absence of bulk motion and of other ways of producing or suppressing particles, the rate of ionisation per unit volume $n_n\Lambda_{ion}$ balances the rate of recombination per unit volume $n_i\Lambda_{rec}$, so that since $n_e = n_i$ for singly ionised ions

$$n_nF\sigma_{ion} = n_e^2\beta \quad \Rightarrow \quad n_e = \left(\frac{n_nF\sigma_{ion}}{\beta}\right)^{1/2}. \tag{2.135}$$

[36] By a factor that is easily shown to be approximately $1/\ln(T/T_F)^{3/2} \ll 1$.

**Cross-sections**

To estimate the order of magnitude of the cross-sections, we consider incident photons of energy $W_{ph} \geq W_{Bohr}$, but still of the order of magnitude of $W_{Bohr}$, because, if the energy of the incident photon is much greater than $W_{Bohr}$, conservation of both energy and impulsion makes the probability of ionisation very small. Hence the electron liberated has the energy $W_{ph} - W_{Bohr} \leq W_{Bohr}$. Likewise, we consider recombining electrons of energy $\leq W_{Bohr}$. Now,

- in order for a photon to produce ionisation, it must (1) pass 'close enough' to an atom, and (2) be absorbed in liberating an electron,
- in order for a free electron to recombine radiatively, it must (1) pass 'close enough' to an ion, and (2) become bound in producing a photon.

In the frame of quantum mechanics, 'close enough' means closer than the quantum uncertainty on the position, that is $\hbar/p$ for a particle of momentum $p$. Therefore,

- for the ionising photon of energy $\sim W_{Bohr}$ and momentum $\sim W_{Bohr}/c$, this distance is $\hbar c/W_{Bohr}$,
- for the recombining electron of momentum $m_e v_{the}$, this distance is $\hbar/m_e v_{the}$.

We deduce the cross-sections for ionisation and recombination

$$\sigma_{ion} \sim \pi \left(\hbar c/W_{Bohr}\right)^2 \times P \tag{2.136}$$

$$\sigma_{rec} \sim \pi \left(\hbar/m_e v_{the}\right)^2 \times P \tag{2.137}$$

where $P$ is the probability of absorption or emission of a photon by the electron during the time uncertainty corresponding to the energy involved, i.e. $\Delta t \sim \hbar/W_{Bohr}$.

We estimate $P$ in a semi-classical way, regarding the bound electron as a harmonic oscillator of angular frequency $\omega = W_{Bohr}/\hbar$ and momentum $\hbar/r_{Bohr}$ so that the velocity is $v \sim \hbar/\left(m_e r_{Bohr}\right) \equiv \alpha c$, where $\alpha$ is the fine structure constant (2.130). For a harmonic oscillator, the speed varies by $\Delta v \sim v$ in a quarter of period, that is the time $\Delta t \sim 1/\omega$, and energy is radiated at a rate given by Larmor's formula as

$$\frac{dW}{dt} = \frac{e^2}{6\pi\epsilon_0 c^3}\left(\frac{dv}{dt}\right)^2. \tag{2.138}$$

Writing $dv/dt \sim \Delta v/\Delta t$ with $\Delta v \sim v \sim \alpha c$ and $\Delta t \sim 1/\omega$, we find that the oscillator radiates during $\Delta t$ the energy

$$\Delta W \sim \frac{dW}{dt} \times \Delta t \sim \frac{e^2\alpha^2\omega}{6\pi\epsilon_0 c}. \tag{2.139}$$

Quantum mechanics tells us that it does so in discrete steps by emitting photons of energy $\hbar\omega$ with the probability $P = \Delta W/\left(\hbar\omega\right)$. From (2.139), this yields

$$P \sim \alpha^3. \tag{2.140}$$

Correct calculations give cross-sections that are roughly three times greater than the values (2.136)–(2.137) with our estimate $P \sim \alpha^3$, i.e.

$$\sigma_{ion} \simeq 10\,(\hbar c/W_{Bohr})^2 \times \alpha^3 \simeq 10^{-21}\ \mathrm{m}^2 \quad \text{(ionisation)} \qquad (2.141)$$

$$\sigma_{rec} \simeq 10\,(\hbar/m_e v_{the})^2 \times \alpha^3 \quad \text{(recombination).} \qquad (2.142)$$

The cross-section (2.141) holds for ionisation of hydrogen by photons of energy $\sim W_{Bohr}$; for heavier atoms, the cross-section is of the same order of magnitude for the liberation of an outer electron. The recombination cross-section (2.142) holds for radiative recombination on the fundamental level of electrons of energy $\leq W_{Bohr}$. The corresponding recombination coefficient is thus given by

$$\beta_{rec} \simeq v_{the}\sigma_{rec} \simeq 10^{-17}/\sqrt{T}\ \ \mathrm{m}^3\,\mathrm{s}^{-1} \qquad (2.143)$$

where $T$ is the electron temperature. We shall use this recombination coefficient to understand why the solar wind is ionised (Section 2.5.7), and the radiative ionisation cross-section to estimate the basic properties of planetary ionospheres (Section 7.1), for deriving comet's properties (Section 7.5) and when studying the interaction of the solar wind with the interstellar medium (Section 8.1).

### 2.4.4 Non-radiative ionisation and recombination

**Ionisation by particle impact**

Another way to produce ionisation is by bombarding with particles. For the impact of a particle to ionise an atom, the kinetic energy of the relative motion of the particle must exceed the ionisation energy $\sim W_{Bohr}$. For an electron of mass $m_e$ this requires $m_e v^2/2 > W_{Bohr}$, i.e. $v > \alpha c$. In this case, $\hbar/m_e v < r_{Bohr}$, so that the effective interaction distance between the electron and the atom (of approximate size $r_{Bohr}$) is no longer $\hbar/m_e v$ but rather $r_{Bohr}$. Hence we expect that the cross-section be of order of magnitude $\pi r_{Bohr}^2$. This holds a fortiori for a particle heavier than an electron.

However, the condition $v > \alpha c$ is not sufficient for producing ionisation, because, for the probability of interaction to be significant, the time of interaction $\Delta t \sim r_{Bohr}/v$ must ensure that the energy $\hbar/\Delta t$ be roughly equal to $W_{Bohr}$, i.e. $v \sim r_{Bohr} W_{Bohr}/\hbar \sim \alpha c$.

For an electron (of mass $m_e$), this requires that the electron kinetic energy be $m_e v^2/2 \sim W_{Bohr}$. However, for an ion or an atom, the mass is $m \gg m_e$, so that the kinetic energy required to ensure $v \sim \alpha c$ is greater by the factor $m/m_e \gg 1$.

Hence in space, impact ionisation is generally produced by electrons of energy of the order of magnitude of $W_{Bohr}$, with an effective cross-section

$$\sigma_{ion} \sim \pi r_{Bohr}^2$$

or by very energetic ions or atoms. With electrons of number density $n$ and temperature $T \sim W_{Bohr}/k_B$, the electron flux is about $n v_{the} \sim n\alpha c$, and the rate of ionisation per atom is $\Lambda_{ion} \sim n v_{the} \times \pi r_{Bohr}^2$. Substituting the electron flux and the Bohr radius (2.129), we find the rate of impact ionisation per atom per electron (in order of magnitude)

$$\Lambda_{ion}/n \sim \frac{\pi \hbar^2}{\alpha m_e^2 c} \sim 2 \times 10^{-14} \text{ m}^3 \text{ s}^{-1}. \qquad (2.144)$$

### Dissociative recombination

We have seen that the cross-section for radiative recombination is extremely small, smaller by many orders of magnitude than the square of the typical atomic size, because of the small probability of photon emission. However, recombination occurs much more easily when the ions can dissociate. Instead of producing a photon, recombination then produces several atoms, by a reaction of the form

$$\text{AB}^+ + e^- \longmapsto \text{A} + \text{B}.$$

The dissociation into several components replaces the emission of a photon to conserve simultaneously the energy and the impulsion. Since no emission of photon is required, the cross-section (in order of magnitude) is given by the value (2.137) with $P = 1$, i.e.

$$\sigma_{rec} \sim \pi \left(\hbar / m_e v_{the}\right)^2 \qquad (2.145)$$

if $\hbar / m_e v_{the} > r_{Bohr}$ i.e. if $m_e v^2/2 < W_{Bohr}$. This yields the coefficient of dissociative recombination

$$\beta = v_{the} \sigma_{rec} \sim 10^{-11}/\sqrt{T} \quad \text{(dissociative recombination)} \qquad (2.146)$$

where $T$ is the electron temperature (assumed smaller than $W_{Bohr}/k_B \sim 1.6 \times 10^5$ K).

This process is more effective than radiative recombination, by a factor of about six orders of magnitude. Hence it is the dominant process when molecular ions are present. In particular, it is the dominant recombination process in the ionospheres of inner planets (Section 7.1.4) and of comets (Section 7.5.2), whose atmospheres are made of complex molecules.

### Charge exchange

Ionisation may also be produced by exchange of an electron between a neutral and an ion, as

$$\text{A}^+ + \text{B} \longmapsto \text{B}^+ + \text{A}$$

in the simple case when the ion is singly ionised. The atom B gives an electron to the ion $\text{A}^+$, and becomes ionised. This does not change the number of ions and free electrons, but changes the chemical nature of the ions and their speeds since the final ion has the properties of the original neutral and vice versa.

If $F_A$ is the incident flux of ions A, the rate of ionisation per atom is $F_A\sigma_{ex}$ where $\sigma_{ex}$ is the cross-section for charge exchange. As for the collision frequency for exchange of momentum between ions and neutrals (Section 2.1), the cross-section is heavily influenced by the electric dipoles induced in the atom and in the ion. For protons and hydrogen atoms with a relative speed of the order of magnitude of the typical solar wind speed, this produces a cross-section greater by two orders of magnitude than the Bohr radius squared, i.e.

$$\sigma_{ex} \simeq 2 \times 10^{-19}\ \mathrm{m}^2. \tag{2.147}$$

This process plays an important role in the heliosphere when the solar wind encounters a large flux of neutrals; we shall see that this holds with neutrals of planetary and cometary origin (Section 7.5.6), as well as with interstellar neutrals (Section 8.1.2), with the interesting further consequence that when the original ion is highly charged, the final ion is left in a highly excited state whose de-excitation produces ultraviolet and X-ray emission.

## 2.5 Problems

### 2.5.1 Linear Debye shielding in a non-equilibrium plasma

In this problem, we generalise the Debye shielding to a non-equilibrium plasma, assuming small perturbations, and prove (2.8)–(2.9).

Consider an electron arriving from infinity (where its velocity is $v$) towards a point charge at the origin that produces the electrostatic potential $\Phi_E(r)$ (with $\Phi_E \to 0$ for $r \to \infty$). Show that if the point charge perturbs weakly the electron (i.e. if $e\Phi_E \ll m_e v^2$), then the electron velocity at distance $r$ is changed by $\delta v$ given by $\delta v/v = e\Phi_E(r)/\left(m_e v^2\right)$.

This velocity change is associated with a perturbation in electron number density around the point charge. For example, if $\Phi_E > 0$ the electrons are attracted and their trajectories are bent toward the charge, which increases their density; since, however, their velocity increases, they spend less time within a given region, which reduces this effect. Show that the net change in electron density is given by $\delta n_e/n = \delta v/v$.

Apply this result to the ions (changing the mass and charge), and deduce that the perturbations in electron and ion densities are given by

$$\delta n_e/n = e\Phi_E(r)\langle v^{-2}\rangle_e/m_e \qquad \delta n_i/n = -e\Phi_E(r)\langle v^{-2}\rangle_i/m_i \tag{2.148}$$

where the brackets denote averages over the velocities at infinity, for electrons and ions respectively.

Deduce the shielding length from Poisson's equation.

Think about the limitations of this calculation.[37] The relation between $\delta n$ and $\delta v$ depends on the symmetry of the problem. How are the results changed with a different geometry? Show that if the point charge is replaced by a

[37] See Meyer-Vernet, N. 1993, *Am. J. Phys.* **61** 249, and references therein.

long wire, then the shielding disappears, whereas with a plane, there is an anti shielding. What happens when one takes into account perturbations that are not small? What kinds of particles are ignored in the above calculation?

**Hints**

We deduce $\delta v/v$ from the conservation of total (electrostatic + kinetic) particle energy between infinity and distance $r$.

To prove that the perturbations in density and speed are related by $\delta n/n = \delta v/v$ (in spherical symmetry), imagine a fictitious sphere of radius $r$ collecting particles arriving on its surface. The particles arriving at grazing incidence with speed $v\left(r\right)$ have an impact parameter $p = r \times v\left(r\right)/v$, from conservation of angular momentum between infinity (where the speed is $v$) and distance $r$. The number of particles collected per second is $nv\pi p^2$. This number is also equal to $n\left(r\right) v\left(r\right) \times \pi r^2$ if the particles have density $n\left(r\right)$ and speed $v\left(r\right)$ at distance $r$ (because for each surface element of the collector, half the particles are incident from one side, and their average perpendicular velocity is $v\left(r\right)/2$). Whence $n\left(r\right)/n = v\left(r\right)/v$.

### 2.5.2 Mean free path in a plasma

Verify from Fig. 2.4, with the plasma properties indicated in the caption, that for the velocity direction to change appreciably, the particle must travel a distance given roughly by (2.22).

### 2.5.3 Particles trapped in a planetary magnetic field

Consider a particle moving along a magnetic line of force near a planet of radius $R_P$, having a dipolar magnetic field (cf. Appendix and Fig. 2.12). Let $\theta_0$ be the pitch angle in the magnetic equatorial plane at distance $r = LR_P$ from the planet, and $B_P$ the magnetic field strength close to the planet in the polar regions. When the particle comes close enough to the planet, it is absorbed because of collisions with the atmosphere. When $L \gg 1$, the line of force crosses the planet surface in the polar regions.

Show that particles can be reflected between the north and south polar regions if

$$\theta_0 \geq \arcsin(B_0/B_P)^{1/2}. \tag{2.149}$$

Calculate this limit angle for $L = 6$.

Show that the bounce motion between the north and south regions follows the equation

$$mv_\parallel^2 + \mu B = \text{constant} \tag{2.150}$$

of a one-dimensional oscillator of potential energy $\mu B$.

Give an order of magnitude of the expression of the three periods of motion of a trapped particle (respectively $T_1$, $T_2$, $T_3$ for gyration, bounce, drift), and show that $T_1 \ll T_2 \ll T_3$. Deduce that, in practice, the adiabatic invariant associated to the gyration (the particle magnetic moment $\mu$) is more invariant than the one associated to the bounce motion, itself more invariant than the one associated to the drift.

With the numerical parameters relevant for the Earth (Appendix), estimate the drift velocity produced by the planet's gravitational field, and show that its ratio to the drift produced by the magnetic gradient is of the order of magnitude of the particle gravitational potential energy to the thermal energy. Deduce that it is in practice completely negligible.

### 2.5.4 Filtration of particles in the absence of equilibrium

In this problem, you will prove a very general result. Let a particle velocity distribution in some region be made of a superposition of Maxwellians of different temperatures. In absence of collisions, the particle velocity distribution in another region where the potential energy of particles is greater has a greater effective temperature. This result holds, for example:

- for the velocity distribution measured on a spacecraft, of particles that are repelled by the spacecraft electrostatic potential (Section 7.2),
- for the environment of a planet or a star, subjected to the body's gravitational potential (and electrostatic field).

Consider a velocity distribution that is a sum of Maxwellians of densities $n_{\alpha 0}$ and temperatures $T_\alpha$, and give a formal expression of its effective temperature $T_0$ (defined from (2.51)).

Show that in the absence of collisions the distribution at a position where the potential energy of the particles has increased by $\Delta\psi > 0$ is again a sum of Maxwellians having the same temperatures $T_\alpha$, but with densities $n_\alpha = n_{\alpha 0} e^{-\Delta\psi / k_B T_\alpha}$. Deduce that the effective temperature is greater than $T_0$.[38]

Prove this result graphically, by redrawing Fig. 2.13 with an initial distribution having more fast particles than a Maxwellian, i.e. whose slope flattens as energy increases.

#### Hints

The effective temperature of a distribution made of a sum of Maxwellians of densities $n_\alpha$ and temperatures $T_\alpha$ is

$$T = \frac{\sum_\alpha n_\alpha T_\alpha}{\sum_\alpha n_\alpha}. \qquad (2.151)$$

A particular application is studied in detail in Section 4.6.

[38] A general analytical proof may be found in Meyer-Vernet, N. 1995, *Icarus* **116** 202.

### 2.5.5 Freezing of magnetic field lines

Consider a bar of copper, of diameter $L$, and imagine you try to put it in a region of strong magnetic field (Fig. 2.17, 1, 2, 3). How long will it take for the magnetic field to penetrate into the bar? Conversely, once the magnetic field has penetrated into the bar, if you try to remove the bar, how long will it take for the magnetic field to disappear from the bar? At what speed should you move the bar for the effects shown on Fig. 2.17 to take place? Figure out the corresponding length and timescales for a cosmic object.

#### Hint

The electric conductivity of copper is about $0.6 \times 10^8$ mho.

### 2.5.6 Alfvén wave

Consider a small-amplitude Alfvén wave propagating along the ambient magnetic field in a uniform plasma at a speed much smaller than the velocity of light. Show that the force produced by the gradient in magnetic pressure is negligible. Show that the electric energy is negligible. Calculate the drift velocity of the particles, and comment.

### 2.5.7 Why is the solar wind ionised?

We have seen that the corona is made essentially of hydrogen, and is so hot that it is virtually completely ionised, and that the solar wind is produced by the expansion of the corona. Use the radiative recombination coefficient to understand why the solar wind remains ionised throughout the heliosphere.

#### Hints

The solar wind density is about $n \sim 5 \times 10^6 \, (d_\oplus/d)^2$ m$^{-3}$ at distance $d$ from the Sun, where $d_\oplus \simeq 1.5 \times 10^{11}$ m is the Sun–Earth distance (1 AU); the electron temperature is about $T \sim 10^5$ K; the size of the heliosphere is of the order of magnitude of $10^2$ AU.

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