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Orbital Resonances and Poynting–Robertson Drag

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Grains orbiting a star experience orbital decay due to Poynting–Robertson drag and can become trapped in commensurability resonances with a planet. We examine conditions for trapping in the circular restricted three-body problem, and derive criteria for trapping in terms of the radiation pressure factor β, planetary mass, resonance location, and the grain's orbital eccentricity. These analytic criteria show good agreement with results of three-body orbital integrations and more elaborate multiplanet simulations by other researchers. We show that resonances under Poynting–Robertson drag are "metastable" because trapped grains inevitably acquire eccentricities large enough to cause their orbits to cross that of the planet. © 1993 Academic Press, Inc.

I. INTRODUCTION

All objects orbiting the Sun are subject to gravitational perturbations by the planets. Small bodies are also acted upon by nongravitational forces. In the early solar system, gas drag in the solar nebula was significant for planetesimals up to at least kilometer size. At the present time, radiation pressure and Poynting–Robertson drag affect interplanetary dust particles of sizes $\lesssim 10^{-2}$ cm. It has been known for some time that the combination of gravitational and nongravitational forces can cause nonintuitive dynamical behavior and produce orbital evolution qualitatively different from that due to either force acting alone. For example, Greenberg (1978) examined effects of damping of eccentricity when a body was in a commensurability resonance with a planet, and showed that the damping resulted in a secular change in semimajor axis that pushed the body's orbit away from the planet. Weidenschilling and Davis (1985) showed that when the resistive medium itself also caused decay of the semimajor axis, then a body could reach an orbit for which the drag and resonant perturbations were in equilibrium; the particle would be trapped in a resonant orbit. They determined limits on size (or gas drag force) that allowed such resonant trapping of planetesimals in the solar nebula. This phenomenon has recently been examined in more detail by Kary *et al.* (1993).

Jackson and Zook (1989, 1992) discovered a similar phenomenon of resonant trapping of small particles in the present solar system. They numerically integrated orbits of grains from asteroidal and cometary sources, subjected to Poynting–Robertson drag. A typical dust orbit would decay until it reached a commensurability resonance with Earth or another terrestrial planet. The grain might then pass through the resonance, or it could be trapped for a period of time ranging from $\sim 10^3$ to more than 10^5 years. While such resonances could be long-lived, they were not permanent; eventually the grain would escape from resonance and its orbit would resume its decay.

Resonant trapping by P-R drag has been studied by other investigators. Marzari *et al.* (1991, 1993) and Marzari and Vanzani (1993) integrated numerous orbits perturbed by as many as five major planets, and developed statistical criteria for trapping of grains of two sizes (15 and 30 μm) in various resonances with Earth. Scholl *et al.* (1992) and Lazzaro *et al.* (1992) have studied the trapping of particles in resonance with a hypothetical planet in the Beta Pictoris disk suggest that resonant trapping might keep the inner part of the disk relatively clear of dust if a planet is present.

These previous investigations have been almost entirely numerical, carried out by integration of huge numbers of particle orbits. Most have involved multiple perturbing planets, on eccentric, inclined orbits. While such cases are more realistic representations of the behavior of dust grains in the real solar system, they tend to obscure the

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basic physical mechanisms that are involved. The goal of the present paper is to elucidate the phenomenon of resonance trapping with P-R drag in the simplest case: the circular restricted three-body problem. In particular, we seek to answer the following questions: What determines whether a grain of a given size passes through a given resonance, or is trapped there? How and why does a trapped particle's orbit evolve with time? Why are P-R resonances only temporary, while gas-drag resonances appear to be stable? Our approach will be mainly analytical, with specific examples from orbital integration to illustrate and test our conclusions.

After this paper was submitted, we received a preprint by Sicardy *et al.* (1993), who have addressed some of these same questions from a somewhat different perspective. Their results and ours appear to be in good agreement.

II. P-R DRAG AND RESONANT PERTURBATIONS

A particle small enough to be affected by Poynting–Robertson drag is also subject to radiation pressure; indeed, the two phenomena are inseparable. The ratio of radiation pressure force to solar gravity, β, is (Burns *et al.* 1979)

$$\beta = \frac{3L_\odot Q_{\rm pr}}{16\pi G M_\odot c \rho r}, \tag{1}$$

where $L_\odot$, $M_\odot$ are the solar luminosity and mass, G the gravitational constant, c the speed of light, and r, ρ are the particle's radius and density. The coefficient $Q_{\rm pr}$ depends on the optical properties of the particle and its size relative to the wavelength of light; a perfectly absorbing particle much larger than the wavelength has $Q_{\rm pr} = 1$. For the examples presented in this paper, we assume that $\beta = 0.285/r$, where r is in micrometers (corresponding to $\rho = 2\text{g cm}^{-3}$ and $Q_{\rm pr} = 1$).

Wyatt and Whipple (1950) derived the rates at which a body in a Keplerian orbit has its semimajor axis a and eccentricity e decay due to P-R drag. Using Eq. (1), we can express these in terms of β:

$$\left[\frac{da}{dt}\right]_{pr} = \frac{-GM_\odot\beta}{ac}\frac{(2 + 3e^2)}{(1 - e^2)^{3/2}} \tag{2}$$

$$\left[\frac{de}{dt}\right]_{pr} = \frac{-5GM_\odot\beta}{2a^2c}\frac{e}{(1 - e^2)^{1/2}}. \tag{3}$$

These equations show that P-R drag always acts to decrease a and e, and that the rate of decay is proportional to β.

A particle is in resonance with a planet when the ratio of their mean motions is the ratio of two small integers. More precisely, we define the resonance variable

$$\phi = (j + 1)\lambda - j\lambda_0 - \tilde{\omega}, \tag{4}$$

where λ is the mean longitude of the particle λ_0 is the mean longitude of the planet (assumed to be on a circular orbit), and $\tilde{\omega}$ is the longitude of the particle's perihelion. The resonance number j is an integer. Then the rate of change of ϕ is

$$\dot{\phi} = (j + 1)n - jn_0 - \dot{\tilde{\omega}}, \tag{5}$$

where n, n_0 are the mean motions of the particle and planet. The geometrical meaning of ϕ is the angle from the particle's pericenter to the longitude of its conjunction with the planet. Outside of resonance, ϕ circulates through 360°, but in resonance ϕ librates about some mean value; generally $\dot{\phi}$ and $\dot{\tilde{\omega}}$ are small compared with n and n_0 so we can assume

$$\frac{n}{n_0} = \frac{j}{(j + 1)}, \tag{6}$$

for an exterior resonance (for an interior resonance, the particle's orbital period is shorter than the planet's, and $n/n_0 = (j + 1)/j$; $\phi = j\lambda - (j + 1)\lambda_0 - \tilde{\omega}$).

Radiation pressure effectively reduces the Sun's attraction for the particle, so

$$n = \left[\frac{GM_\odot(1 - \beta)}{a^3}\right]^{1/2}, \tag{7}$$

implying that in resonance

$$\frac{a}{a_0} = (1 - \beta)^{1/3}\left[\frac{j + 1}{j}\right]^{2/3}. \tag{8}$$

As β increases, the particle's semimajor axis decreases. Exterior resonances are shifted closer to the planet's orbit, while interior resonances are farther from the planet (closer to the Sun). This effect changes the strength of a given resonance for different-sized particles.

We will consider explicitly only exterior resonances, which are the only ones that can produce long-lived trapping. The resonant perturbations are given by (Greenberg 1973)

$$\left[\frac{da}{dt}\right]_{\rm Res} = -(j + 1)aen\mu C_j \sin\phi \tag{9}$$

$$\left[\frac{de}{dt}\right]_{\text{Res}} = \frac{-\mu n}{2} C_j \sin\phi, \tag{10}$$

where μ is the mass of the perturbing planet normalized to $M_\odot$. C_j is a Laplace coefficient that depends on the resonance number j and on the ratio of semimajor axes of the particle and planet. The values of C_j can be calculated from the series expansion of the disturbing function as described by Brouwer and Clemence (1961), using the value of a/a_0 from Eq. (8). As mentioned above, the effect of radiation pressure shifts the positions of the resonances. Higher β means that exterior resonances are closer to the planet and resonant perturbations stronger; C_j is thereby increased. For interior resonances, the resonances are farther from the planet and correspondingly weaker, and C_j is smaller for a given j. Resonance positions (in units of the planet's orbital radius a_0) and Laplace coefficients for different values of β are given in Table I. Note that for sufficiently large β, some of the "exterior" resonances are located at $a < 1$; i.e., a particle with semimajor axis less than the planet's can still have a smaller mean motion. Our analysis loses its validity in that case.

III. CRITERIA FOR RESONANT TRAPPING

If we assume that a particle is trapped in resonance, such that a is constant, then the orbital decay due to P-R drag is balanced by resonant perturbations, i.e., $[da/dt]_{\text{PR}} + [da/dt]_{\text{Res}} = 0$. From Eqs. (2) and (9), this implies

$$\sin\phi = \frac{-(GM_\odot/a_0)^{1/2}}{\mu c} \frac{\beta}{(1-\beta)^{2/3}} \frac{j^{1/3}}{C_j(j+1)^{4/3}} \frac{(2+3e^2)}{e(1-e^2)^{3/2}} \tag{11}$$

where we have assumed that the particle's semimajor axis is given by Eq. (8). $[da/dt]_{\text{PR}}$ is always negative, so in order for the resonant torque to counteract PR drag, $\sin\phi$ must be negative. The geometrical definition of ϕ implies that for $\phi < 0$, conjunctions occur after the particle's aphelion.

The maximum resonant torque occurs for $\sin\phi = -1$. This corresponds to a value of β such that

$$\frac{\beta}{(1-\beta)^{2/3}} < \frac{\mu c}{(GM_\odot/a_0)^{1/2}} \frac{C_j(j+1)^{4/3}}{j^{1/3}} \frac{e(1-e^2)^{3/2}}{(2+3e^2)}. \tag{12}$$

Larger values of β (smaller particles) make P-R drag too strong to be counteracted by the resonant torque. Likewise, for a given value of β, e must exceed a critical value such that

$$\frac{e(1-e^2)^{3/2}}{2+3e^2} \geq \frac{(GM_\odot/a_0)^{1/2}}{\mu c} \frac{\beta}{(1-\beta)^{2/3}} \frac{j^{1/3}}{C_j(j+1)^{4/3}}. \tag{13}$$

If terms of order e^2 are neglected, the minimum value of e for trapping is

$$e_{\min} \simeq \frac{2(GM_\odot/a_0)^{1/2}}{\mu c} \frac{\beta}{(1-\beta)^{2/3}} \frac{j^{1/3}}{C_j(j+1)^{4/3}}. \tag{14}$$

If terms through e^2 are retained, Eq. (13) can be expressed as a quadratic and solved for $e_{\min}$; the values are slightly higher than given by (14), but the differences are significant only for small j.

The requirement that e be nonzero is implicit in Eq. (9) where the resonant perturbation of a is proportional to e. A physical explanation was given by Weidenschilling and Davis (1985): If conjunction occurs after the grain's aphelion ($\sin\phi < 0$), then the closest approach occurs after conjunction. The grain lags slightly behind the planet, so the planet's gravitational pull has a component in the forward direction. This impulse gives an increase in the grain's orbital energy, compensating for the energy lost due to P-R drag between conjunctions.

From Eq. (14) we can see that for a given resonance, a smaller grain (higher β) needs a higher e to be trapped (the dependence of C_j on β is not sufficient to counteract the factor $\beta/(1-\beta)^{2/3}$). Also, a less massive planet (smaller μ) has a higher threshold $e_{\min}$ for trapping. These results are consistent with the findings of Jackson and Zook (1989) that particles with radii less than 60 μm were not trapped into resonance with Mars, while Earth and Venus often trapped grains with $r > 10$ μm. The factor $C_j(j+1)^{4/3}$ in the denominator implies that for a given β, $e_{\min}$ is less, and trapping easier, at higher values of j, as found in the numerical study by Marzari *et al.* (1993). Thus, a grain might pass through one or more resonances before being trapped. One limit to this process is the fact that for sufficiently large j (and β), the grain's perihelion distance $q(e_{\min}) = a(1 - e_{\min})$ will be less than a_0, allowing close encounters with the planet before a stable resonance is possible. The derivation of $e_{\min}$ assumes $\sin\phi = -1$ and therefore provides only a lower limit on the required eccentricity; larger values of e allow a larger range of ϕ that can sustain the resonance. These conditions ($e > e_{\min}$, $q > a_0$) explain the finding of Marzari *et al.* (1993) that the maximum probability of trapping into a given resonance occurs for nearly tangent orbits, i.e., the highest values of e that do not allow close encounters with the planet. In their simulations capture into resonance is rare unless e is several times $e_{\min}$. This is probably due to their choice of a substantial initial inclination of 10°, while Eq. (14) assumes that inclination is small. Table II gives values of $e_{\min}$ and $q(e_{\min})$ for $\beta = 0.01$ and 0.1,

TABLE I
Resonance Locations and Laplace Coefficients Exterior Resonances

J	$\beta = 0$ a_{res}	$\beta = 0$ C_j	$\beta = 0.01$ a_{res}	$\beta = 0.01$ C_j	$\beta = 0.1$ a_{res}	$\beta = 0.1$ C_j
1	1.5874	0.857	1.5821	0.894	1.5326	1.253
2	1.3104	4.968	1.30609	5.029	1.2652	5.673
3	1.2114	6.567	1.2074	6.688	1.1696	8.048
4	1.1604	8.167	1.1565	8.371	1.1204	10.773
5	1.1292	9.769	1.1255	10.075	1.0903	13.910
6	1.1082	11.372	1.1045	11.802	1.0700	17.541
7	1.0931	12.975	1.0895	13.550	1.0554	21.772
8	1.0817	14.578	1.0781	15.321	1.0444	26.748
9	1.0728	16.182	1.0692	17.113	1.0357	32.663
10	1.0656	17.785	1.0620	18.929	1.0288	39.799
11	1.0597	19.389	1.0562	20.767	1.0232	48.572
12	1.0548	20.992	1.0513	22.628	1.0184	59.634
13	1.0507	22.596	1.0471	24.512	1.0144	74.079
14	1.0471	24.200	1.0436	26.421	1.0109	93.914
15	1.0440	25.803	1.0405	28.353	1.0079	123.326
16	1.0412	27.407	1.0378	30.310	1.0053	172.923
17	1.0388	29.011	1.0354	32.291	1.0030	280.742
18	1.0367	30.615	1.0332	34.298	1.0009	786.288

$\mu = 3 \times 10^{-6}$, corresponding to an Earth-mass planet. Note that for some combinations of j, β, and μ, capture is impossible for any value of e (e.g., $j = 1$, $\beta = 0.1$ in Table II).

One should note that $e > e_{\min}$ is necessary, but not sufficient, for capture into resonance. If $\sin \phi > 0$, then the resonant perturbations will add to the P-R orbital decay and drive the grain through the resonance. Approach to, or passage through, a resonance will usually change the grain's orbital eccentricity. If e is initially small, then da/dt is dominated by P-R drag (cf. Eq. (2)), while de/dt is due to resonant perturbations (Eq. (10)). Since $(da/dt)_{PR}$ varies only as $1/a$, the time for a grain to pass through successive resonances will be comparable. From Eq. (10) we can expect e to change by an amount proportional to $C_j \sin \phi$ during each resonance passage. Since C_j increases with j, e performs a random walk with steps of increasing magnitude at each resonance. If e exceeds $e_{\min}$ and $\sin \phi < 0$ at some resonance (before the orbits cross and the grain has a close encounter with the planet), the grain will be trapped there. This behavior is seen in Fig. 1; where a grain with initial $e = 0$ and a just

TABLE I—*Continued*

J	$\beta = 0$		$\beta = 0.01$		$\beta = 0.1$	
	a_{res}	C_J	a_{res}	C_J	a_{res}	C_J
18	.9646	30.615	.9614	27.473	.9313	11.422
17	.9626	29.011	.9594	26.188	.9294	11.373
16	.9604	27.407	.9572	24.887	.9273	11.295
15	.9579	25.803	.9547	23.571	.9248	11.183
14	.9551	24.200	.9519	22.238	.9221	11.036
13	.9518	22.596	.9486	20.888	.9190	10.848
12	.9480	20.992	.9449	19.522	.9153	10.615
11	.9436	19.389	.9405	18.139	.9111	10.332
10	.9384	17.785	.9353	16.739	.9061	9.995
9	.9322	16.182	.9291	15.322	.9000	9.596
8	.9245	14.578	.9214	13.887	.8926	9.130
7	.9148	12.975	.9118	12.435	.8833	8.587
6	.9023	11.372	.8993	10.965	.8712	7.961
5	.8856	9.769	.8826	9.477	.9550	7.240
4	.8618	8.167	.8589	7.971	.8320	6.414
3	.8255	6.567	.8227	6.448	.7970	5.470
2	.7631	4.968	.7606	4.908	.7368	4.395
1	.6300	0.857	.6279	.820	.6082	0.473

outside the 3/4 resonance passes through resonances with $j = 3$ through 8 before being trapped in the (9/10) resonance. This example and the others in this paper were produced by numerical integration of orbits using the Bulirsch-Stoer algorithm from Press *et al.* (1986).

If a grain's orbital eccentricity is sufficiently large before it approaches a resonance, then $\tilde{\omega}$ will not be changed very much by the perturbations. The value of ϕ at the time of resonance passage will then depend only on the difference in the particles' and planets' mean longitudes. In that case, the probability of trapping into a given resonance should be roughly proportional to the range of ϕ that allows the resonant torque to counteract the P-R drag; this probability should not exceed 0.5. However, when e is small, $\tilde{\omega}$ can vary during resonance passage. This variation will change the value of ϕ and affect the trapping probability; its dependence on e, β, j, and μ remains to be explored.

IV. EVOLUTION WHILE IN RESONANCE

Numerical integrations of resonant orbits show that after a particle is trapped, a is essentially constant while e increases with time. If we take the "equilibrium" value of $\sin \phi$ from Eq. (11) and substitute it into Eq. (10),

TABLE II
Minimum Eccentricities for Capture into Resonance ($\mu = 3 \times 10^{-6}$)

	$\beta = 0.01$			$\beta = 0.1$		
j	a_{res}	e_{min}	q_{emin}	a_{res}	e_{min}	q_{emin}
1	1.5821	.3506	1.027	1.5326	1.0000	0.000
2	1.3060	.0387	1.255	1.2652	.5032	0.628
3	1.2074	.0227	1.180	1.1696	.2144	0.919
4	1.1565	.0148	1.139	1.1204	.1253	0.980
5	1.1255	.0104	1.114	1.0903	.0809	1.002
6	1.1045	.0077	1.096	1.0700	.0552	1.011
7	1.0895	.0059	1.083	1.0554	.0391	1.014
8	1.0781	.0046	1.073	1.0444	.0284	1.015
9	1.0692	.0038	1.065	1.0357	.0210	1.014
10	1.0620	.0031	1.059	1.0288	.0157	1.013
11	1.0562	.0026	1.053	1.0232	.0118	1.011
12	1.0513	.0022	1.049	1.0184	.0089	1.009
13	1.0471	.0019	1.045	1.0144	.0067	1.008
14	1.0436	.0016	1.042	1.0109	.0049	1.006
15	1.0405	.0014	1.039	1.0079	.0035	1.004
16	1.0378	.0013	1.036	1.0053	.0024	1.003
17	1.0354	.0011	1.034	1.0030	.0014	1.002
18	1.0332	.0010	1.032	1.0009	.0005	1.000

$$\frac{de}{dt} = \frac{GM_\odot}{2a_0^2 c} \frac{\beta}{(1-\beta)^{2/3}} \left(\frac{j}{j+1}\right)^{4/3} \left[\frac{(2+3e^2)}{(j+1)e(1-e^2)^{3/2}} - \frac{5e}{(1-e^2)^{1/2}}\right]. \quad (15)$$

For given values of β and j, the term outside the brackets is constant; call this A. After some manipulation and elimination of terms higher than e^2, we have

$$2e\frac{de}{dt} = \frac{d(e^2)}{dt} \simeq 2A\left[\frac{2}{(j+1)} - 5e^2\right]. \quad (16)$$

Neglecting the e^2 term, Eq. (16) implies $e \propto t^{1/2}$, as found by Sicardy *et al.* (1993). However, retaining the e^2 term yields the solution

$$e(t) = \left\{\frac{2}{5(j+1)}(1 - \exp[-(t-t_0)/\tau])\right\}, \quad (17)$$

where t_0 is the time at which trapping occurs. The time constant τ is equal to $1/10A$. For the parameters used to produce Fig. 1, $\tau \simeq 6200$ years. Equation (17) predicts $e = 0.077$ at $t - t_0 = 10^3$ years, in excellent agreement

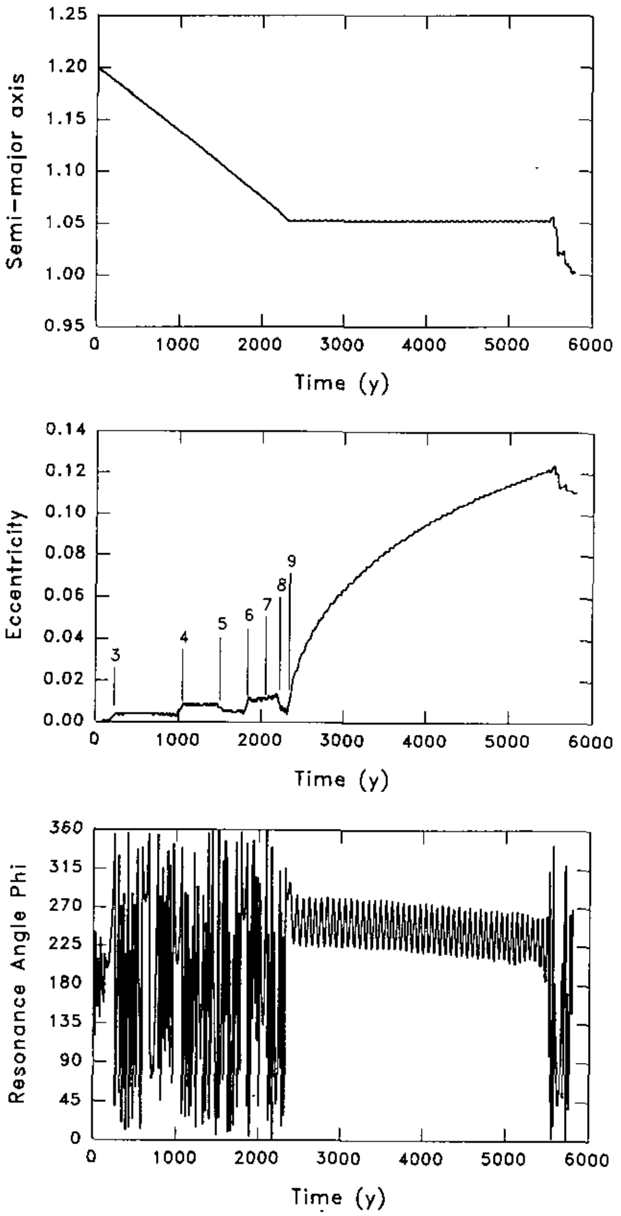

FIG. 1. Semimajor axis, eccentricity, and resonance angle for a 10-μm-diameter (β = 0.057) particle started in a circular orbit at 1.2 AU with i = 0°. The particle passes through resonances j = 3 through 8 (labeled on the e vs t plot), each causing a step in e. It is trapped in the 9/10 resonance for about 3000 years until a close encounter with the planet removes it from resonance.

with the actual value. Equation (17) also predicts that e approaches a limiting value as t becomes large:

$$e_{\max} = \left[\frac{2}{5(j+1)}\right]^{1/2}. \quad (18)$$

The maximum eccentricity depends only on the resonance number j, and is independent of β or μ. This behavior is similar to that found by Weidenschilling and Davis (1985) for resonances in the solar nebula with gas drag, where the maximum eccentricity depended only on j and the nebular structure. However, there is one important difference. Gas drag is more effective in damping eccentricity than is P-R drag. For plausible nebular models, the maximum eccentricity attained in resonances with gas was significantly lower, so that the orbit of the planetesimal never crossed that of the planet. For P-R drag, $e_{\max}$ is always large enough for the orbits to cross. Table III shows the minimum perihelion distance $q_{\min} = a(1 - e_{\max})$ for a 20-μm particle (β = 0.0285). While $e_{\max}$ is independent of β, a decreases for larger β, so smaller particles would have even smaller perihelia. Note that the perihelia in Table III fall within a narrow range for all values of j, even though their aphelia are spread over a much wider range.

There are some caveats to our analysis. The values of $e_{\max}$ are large enough so that our neglect of terms higher than e^2 may not always be valid. Also, our assumption that while in resonance ϕ is an "equilibrium" value given by Eq. (11) is not strictly correct. In fact, ϕ can librate about this value, sometimes with quite large amplitude. The oscillations in ϕ are primarily due to changes in the mean motion as a librates about the exact resonance. There is also a secular decrease in the mean magnitude of sin ϕ as e increases, as implied by Eq. (11). The large e's and librations of ϕ tend to destabilize the resonances. In most cases, the grain is eventually removed from resonance by a close encounter with the planet after e grows large enough to allow the orbits to cross. More rarely, the libration amplitude simply increases until ϕ circulates, and the particle leaves the resonance. Figure 2 shows an example of a marginally stable resonance with a large libration amplitude. The grain is caught in the j = 8 resonance for about 1500 years, with ϕ varying over a range of 120°. The libration in ϕ is accompanied by oscillations in a and e. The particle's orbit becomes planet-crossing about 500 years after entering the resonance, but it avoids close approaches for another 1000 years. Eventually the increasing e and librations of ϕ allow a close encounter, which removes it from resonance and decreases a. Then the particle is briefly trapped in the j = 10 interior resonance. Here ϕ librates around ≃330°, meaning that conjunctions occur before the particle's perihelion. The planets' perturbations add angular momentum, giving $da/dt > 0$, but $de/de < 0$, and e quickly falls below the value needed to maintain a constant semimajor axis. For this reason, interior resonances tend to be short-lived.

Figure 3 shows a 20-μm particle in an unusually long-lived resonance. Fortuitously, the particle enters the j = 8 resonance with ϕ very close to the equilibrium value, and a small libration amplitude. However, the libration amplitude slowly increases with time, and continues to

TABLE III
Maximum Eccentricity and Corresponding Perihelion and Aphelion

j	a_{res}*	e_{max}	$q(e_{max})$	$Q(e_{max})$
1	1.572	0.447	0.869	2.275
2	1.298	0.365	0.824	1.772
3	1.200	0.316	0.820	1.579
4	1.149	0.283	0.824	1.474
5	1.118	0.258	0.830	1.406
6	1.098	0.239	0.835	1.360
7	1.083	0.224	0.841	1.326
8	1.071	0.211	0.845	1.297
9	1.062	0.200	0.850	1.274
10	1.055	0.191	0.854	1.257
11	1.050	0.183	0.858	1.242
12	1.045	0.175	0.861	1.228

*Resonance position corresponds to β = 0.0285, particle diameter 20 μm.

do so after e reaches a constant value. After more than 10^5 years in resonance, the particle is removed from resonance by a close encounter with the planet.

V. CONTRAST WITH GAS DRAG

Poynting-Robertson resonances appear to be "metastable;" our orbital integrations yielded no example of a permanently stable resonance. While this does not prove that stable orbits do not exist, we suspect that they do not. Even in the circular restricted three-body problem, any stable resonant orbits must compose a statistically insignificant fraction of orbital parameter space. In the real solar system with a plurality of perturbing planets, stable resonant orbits are even less plausible. This instability is due to the relatively ineffective eccentricity damping by P-R drag, which inevitably leads to crossing orbits. In contrast, resonant orbits under the influence of gas drag are generally stable. The greater damping by gas leads to $e_{max} \simeq [(\Delta V/V_K)/(j+1)]^{1/2}$, where $(\Delta V/V_K)$ is the fractional deviation of the gas from Keplerian rotation (Weidenschilling and Davis 1985). Since plausible models of the solar nebula have $(\Delta V/V_K) \lesssim 10^{-2}$; this gives e_{max} about an order of magnitude lower than that given by Eq. (18). Because of the low e_{max} (and the fact that resonances are not shifted radially as they are by radiation pressure), resonant orbits with gas drag never cross the orbit of the planet.

The resonant perturbations increase in strength with j as shown by the factor $(j+1)C_j$ in Eq. (9). In the gas drag case, this implied that any drag force, however strong, could be balanced in principle by some resonance with j sufficiently large. The actual upper limit to the drag (or lower limit to the size of a planetesimal) that allowed trapping was set by the onset of resonance overlap at $j \simeq 0.5\mu^{-2/7}$ (Wisdom 1980). For Earth ($\mu = 3 \times 10^{-6}$), $j_{max} \simeq 18$. For P-R drag, the maximum value of j is limited by radiation pressure. As β is increased, the resonance locations are shifted inward, so that resonances with j above some value are located inside the planet's orbit. Setting $a = a_0$ in Eq. (8) we find

$$j_{max} = \frac{(1-\beta)^{1/2}}{1-(1-\beta)^{1/2}}. \tag{19}$$

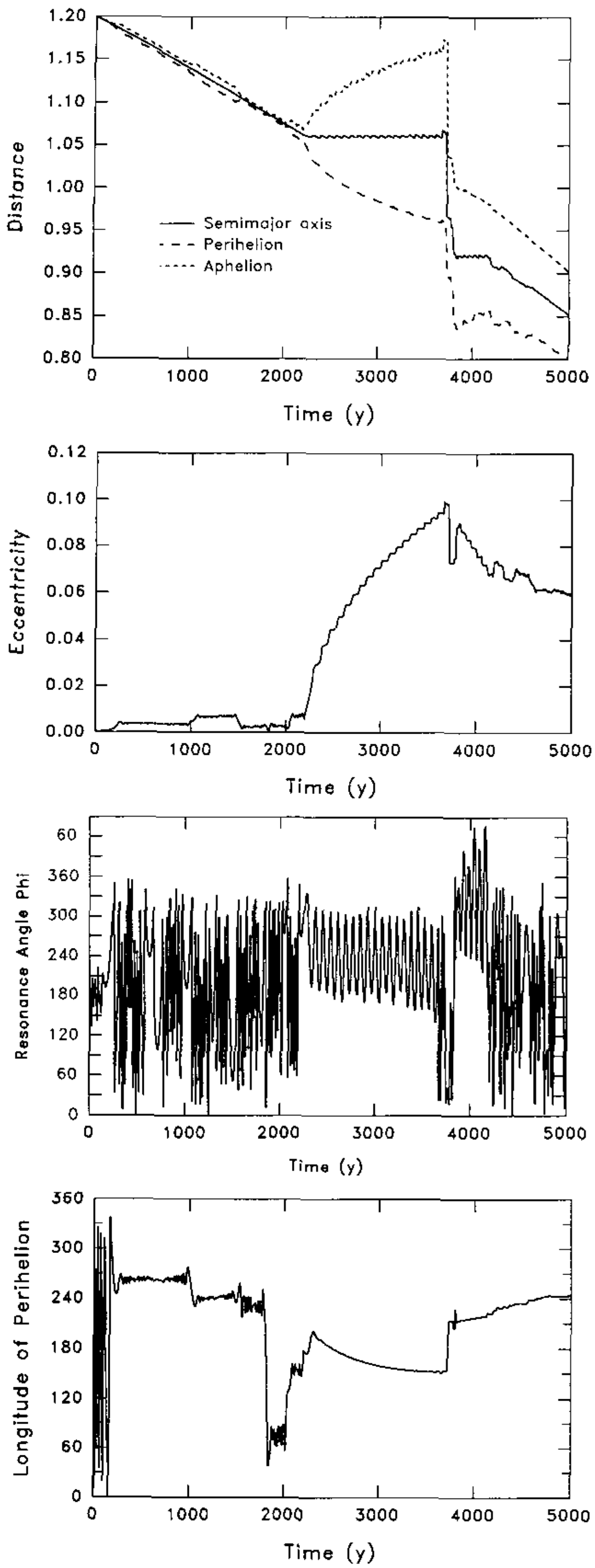

FIG. 2. An example of a short-lived resonance with a large libration amplitude. The 10-μm (β = 0.057) particle was started with the same initial conditions as in Fig. 1, except the true anomaly differed by 90°. It is captured into the 8/9 resonance; a and e oscillate in phase with ϕ. Two close encounters lower a, and the particle is briefly trapped in the 11/10 interior resonance. In this resonance, a is constant while e decreases; ϕ librates around a mean value near 330°.

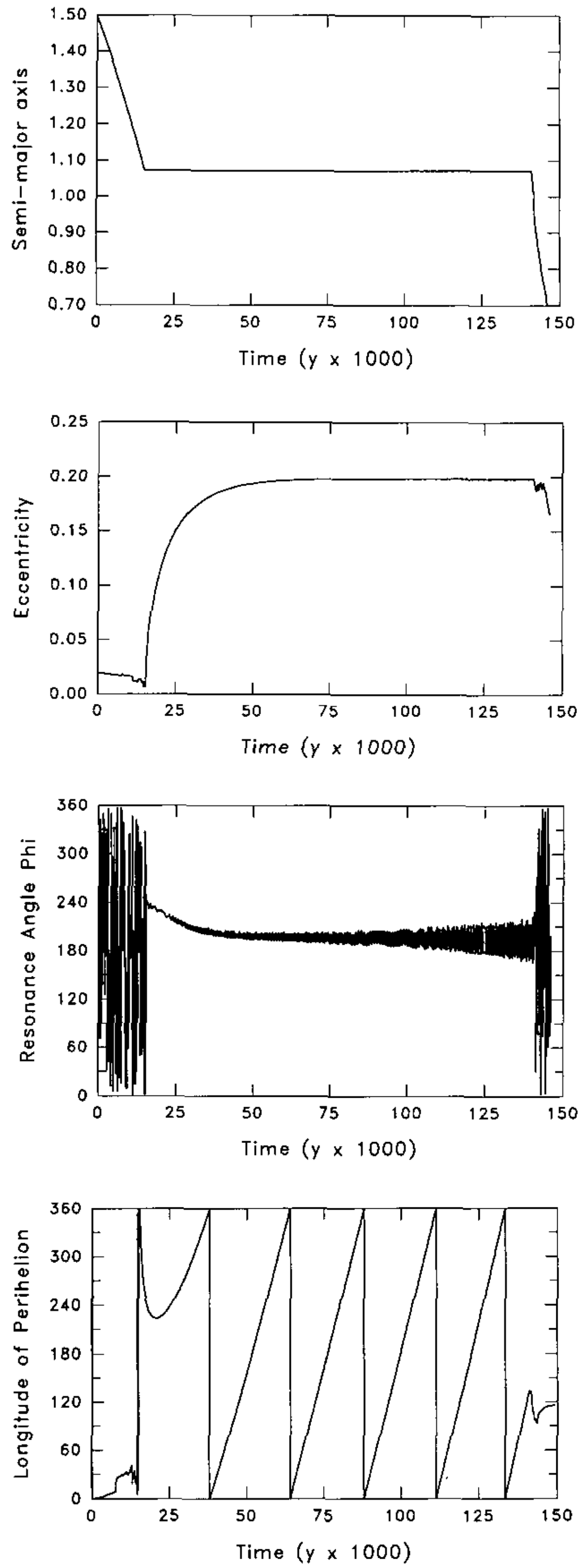

FIG. 3. An example of an unusually long-lived resonance. The 20-μm particle was started at 1.5 AU with initial e = 0.02, i = 1°. The libration amplitude of ϕ is initially small, but grows slowly, removing the grain from resonance after more than 10^5 years. During this time, $\tilde{\omega}$ circulates slowly with period of about 25,000 years.

For $\beta = 0.1$, $j_{max} = 18$ (independent of μ). However, in most cases of interest, resonant capture is limited not by resonance overlap or resonance shifting, but by close approaches to the planet.

Another apparent difference is the existence of a definite threshold e_{min} for capture into resonance in the P-R case (cf. Eq. (14)). In the gas drag case, Weidenschilling and Davis (1985) and Kary *et al.* (1993) found that a planetesimal with $e = 0$ approaching a resonance was invariably captured, unless the drag parameter exceeded a critical value that did not depend on e. In fact, since the resonant perturbation of semimajor axis is the same for gas drag and P-R drag (cf. Eq. (9)), there must be some finite eccentricity for trapping to occur in either case. However, e_{min} is generally much lower for the gas drag case; one can show that when the drag parameter is the maximum value that allows trapping, $e_{min} \approx 2(\Delta V/V_K)[j/(j+1)]^{2/3}$. This value is small enough that in all of the examples given by Weidenschilling and Davis (1985), the approach to a given resonance was sufficient to increase e and allow trapping at that resonance. In the P-R case, $e_{min} \propto \beta$ (Eq. (14)), so we can expect that sufficiently large particles (with small β) could be captured into resonance from initially circular orbits. However, the orbital evolution of such objects under P-R drag will be very slow, and in the real solar system, gravitational perturbations would control their evolution.

V. DISCUSSION

We have shown that the possibility of trapping a grain into resonance with a planet depends on the combination of parameters β, μ, j, and e. These set necessary conditions for trapping. It is also required that the relative positions of grain and planet yield the appropriate sign of the resonance angle ϕ at the time of resonance passage; this condition introduces a stochastic element into the capture process. In general, the peak eccentricity e_{max}, and sometimes the threshold value e_{min}, are large enough so that crossing orbits and close approaches to the planet can inhibit capture and aid escape from resonance.

The numerical examples presented here are based on the parameters of the Sun–Earth system. However, our analysis applies equally to other systems, such as Beta Pictoris. This A-type star has a higher ratio of luminosity to mass, so a given β corresponds to particles a few times larger. The metastable character of the resonances and their limited lifetime means that a planet is not a very effective barrier to P-R decay of orbiting dust particles. This conclusion is robust, because e_{max} is independent of the mass of any planet in the Beta Pictoris system. Orbital lifetimes of individual particles will be extended, but the long-term mean radial mass flux will not be affected significantly by resonances alone. At best, resonances can cause an increased concentration of dust near a planet's orbit (and extending slightly inside it), but will not produce a clear "hole" in the inner part of a dust disk. Other processes acting in concert with resonances may act to produce such clearing, e.g., collisional destruction of grains while they are trapped in eccentric resonant orbits, combined with expulsion of fragments by radiation pressure. The effectiveness of such a process will depend on many factors, such as the size distribution of grains, their impact strength, and the density of the swarm, and will require more elaborate simulations than this analysis.

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Note added in proof. Roques *et al.* (1993) have identified another loss mechanism for grains. A trapped grain is usually removed from resonance by a close encounter with the planet. If the velocity perturbation due to that encounter is large enough, the grain may be placed into a hyperbolic orbit or a short-lived orbit with small periastron distance. Their simulations produced substantial clearing inside the orbit of a hypothetical planet of $5M_{\oplus}$ located 20 AU from Beta Pictoris. The probability of a large orbital deflection in an encounter depends on the planet's mass and orbital radius, or more precisely its escape velocity and orbital velocity (Weidenschilling 1975). A terrestrial-type planet deep within a star's gravitational well is not effective at ejecting grains from the system, and thus our simulations did not show this behavior. A massive planet in a larger orbit can be an effective barrier and clear a hole in a dust disk. I thank B. Sicardy and H. Scholl for discussions on this point.

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