# STATIONARY CONFIGURATIONS FOR CO-ORBITAL SATELLITES WITH SMALL ARBITRARY MASSES

STÉFAN RENNER and BRUNO SICARDY
*LESIA, Observatoire de Paris, 92195 Meudon Cédex, France, e-mail: stefan.renner@obspm.fr*

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**Abstract.** We derive general results on the existence of stationary configurations for $N$ co-orbital satellites with small but otherwise arbitrary masses $m_i$, revolving on circular and planar orbits around a massive primary. The existence of stationary configurations depends on the parity of $N$. If $N$ is odd, then for any arbitrary angular separation between the satellites, there always exists a set of masses (positive or negative) which achieves stationarity. However, physically acceptable solutions ($m_i > 0$ for all $i$) restrict this existence to sub-domains of angular separations. If $N$ is even, then for given angular separations of the satellites, there is in general *no* set of masses which achieves stationarity. The case $N = 3$ is treated completely for small arbitrary satellite masses, giving all the possible solutions and their stability, to within our approximations.

**Key words:** central configurations, co-orbital satellites, $N$ body problem, relative equilibria

## 1. Introduction

Configurations with a massive central body surrounded by small co-orbital satellites are found in several instances in the Solar System. Examples are found in the Saturnian system: a satellite (Helene) librates near the $L_4$ point of Dione, and two satellites, Telesto and Calypso, librate near the $L_4$ and $L_5$ of Tethys, respectively, while the co-orbital satellites Janus and Epimetheus oscillate in horseshoe orbits around their mutual $L_3$ point.

In another context, the four co-orbital Neptune ring arcs might be explained, at least partly, by the presence of several hypothetical co-orbital satellites which would confine the observed dusty ring material (Renner and Sicardy, in preparation). More generally, a ring close to the Roche zone of its planet might evolve, through accretion, into a collection of $N$ co-orbital satellites which gather most of the mass of the ring material.

Our aim in this paper is to derive some general results on *stationary planar configurations for N co-orbital satellites* orbiting a much more massive central planet (planar $1 + N$ body problem). These stationary configurations are sometimes called *relative equilibria*, that is, special configurations of masses of the $N$ body problem which rotate rigidly, with a constant angular velocity about their center of mass, if given the correct initial momenta. In rotating coordinates these special solutions become fixed points, hence the name relative equilibria. The

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relative equilibria are solutions of the planar $N$ body problem resulting from an appropriate choice of initial velocities of a *central configuration*. Central configurations are the configurations of the $N$ body problem such that the total newtonian acceleration on every body is equal to a constant multiplied by the position vector of this body, the center of mass of the configuration being taken as the origin. Such configurations give rise to simple, explicit solutions of the $N$ body problem (homothetic collapse if the bodies are released with zero initial velocity, relative equilibria, ...).

To within our approximations, detailed herein, we generalize the work by Salo and Yoder in 1988 (hereafter referred to as SY88) to the case of satellites with small but arbitrary (not necessarily equal) masses. Indeed, in SY88, only the stationary configurations of $N$ ($N \leqslant 9$) *identical* co-orbital satellites are mapped out. This work was itself an extension of Maxwell's study (1859) on the stability of Saturn's rings in his Adams prize essay of 1856. This latter work was restricted to symmetrically spaced identical many particles.

However, other papers concerning relative equilibria with a dominant mass, stayed fairly independent from SY88 because of the use of the different terminology mentioned above. Hall (1987) also studied the planar relative equilibria of the $1 + N$ body problem where one mass is large and the other $N$ masses are small and equal. He found that, when $N$ is sufficiently large, the only possible relative equilibrium is *Maxwell's ring*, that is, a regular $N$-gon with a central mass, and that other configurations are possible for small $N$. In particular, he solved the problem of 1 large $+$ 3 small identical bodies. Moeckel (1994) found a necessary and sufficient condition for the linear stability of relative equilibria of the $1 + N$ body problem with $N$ small but not necessarily equal masses. He showed that Maxwell's ring is linearly stable if and only if $N \geqslant 7$. A bifurcation value $M_{\mathrm{bif}}$ was found such that this configuration is linearly stable if and only if the central mass $M > M_{\mathrm{bif}}$ (Roberts, 2000).

In a related but different vein, Albouy (1996) recently classified all the relative equilibria for four equal masses. There are essentially only four configurations: a collinear solution, a square, an equilateral triangle with a body at the center and an isosceles triangle with a body on the axis of symmetry. No complete classification exists for four unequal masses or for the $N$ body problem with $N \geqslant 5$. In the 5-body problem, Roberts (1999) found a one-parameter family of degenerate relative equilibria which includes a negative mass, and thus concluded that the masses being positive is a necessary condition for the set of relative equilibria equivalence classes to be finite. Albouy and Llibre (2002) investigated the 3-D configurations for 1 large $+$ 4 small identical bodies. No stationary configurations are possible in the non-planar case, but homothetic central configurations are possible. This 3-D situation will not be considered here.

What we show in this paper is that the existence of possible planar stationary solutions of the $1 + N$ body problem depends on whether $N$ is odd or even. Namely, if $N$ is odd, and for a given angular configuration, there is always a set of satellite masses (although not always physically meaningful, as some of them may be negative) which achieves stationarity. For $N$ even on the contrary, there is in general *no* combination of masses which ensures stationarity for a given angular configuration.

We give some general results on the linear stability of the stationary configurations, and we treat completely the case of $N = 3$ satellites with arbitrary masses.

We finally present numerical methods to derive the possible stationary configurations for given satellite masses, or conversely, to derive the satellites masses, given a stationary configuration.

## 2. Equations of Motion

Consider the $1 + N$ body problem with $N$ co-orbital satellites moving with the same average mean motion $n_0$ and the same average orbital radius $r_0$ around a point mass $M$.

We assume that all the satellites lie on the same plane (planar problem), and that the satellite masses are small with respect to the central mass $M$: for all $i = 1, \ldots, N$, $0 \leqslant m_i/M \ll 1$, where $m_i$ is the mass of the $i$-th satellite. A more rigorous way to define the $N$ infinitesimal masses is the following: for all $i = 1, \ldots, N$, $m_i = \epsilon\mu_i$, where $\mu_i \in I\!R^+$ and $\epsilon > 0$ is a small parameter that tends to zero.

We write the equations of motion in a coordinate system with origin at $M$ and rotating with the angular velocity $n_0$. We assume that the orbital eccentricity of each satellite is zero and that any two satellites $i$ and $j$ never get closer than a few mutual Hill sphere radii $r_{\mathrm{H}} = r_0[(m_i + m_j)/M]^{1/3}$. This avoids chaotic motion and ensures that the satellites remain co-orbital. In particular, we will not consider here the stationary points aligned with the satellite and the planet (like the $L_1$ or $L_2$ Lagrange points of the $N = 2$ problem).

The motion is described by the coordinates $(\phi_i, \xi_i)$, $i = 1, \ldots, N$, where $\phi_i$ is the longitude of satellite $i$ with respect to an arbitrary origin, and $\xi_i = \Delta r_i/r_0$ is the relative radial excursion of that satellite with respect to the average radius $r_0$ (see Figure 1).

For convenience and with no loss of generality, we assume that $n_0 = 1$, $r_0 = 1$ and $M = 1$. Classical derivations based on the above assumptions then show (see e.g. SY88) that the equations of motion are:

$$\dot{\phi}_i = -\frac{3}{2}\xi_i, \qquad \dot{\xi}_i = -2\sum_{j\neq i} m_j f'(\phi_i - \phi_j), \tag{1}$$

where

$$f(\phi) = \cos\phi - \frac{1}{2|\sin\phi/2|}, \tag{2}$$

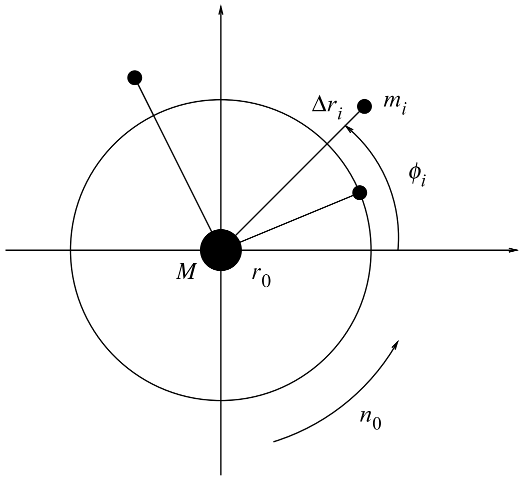


*Figure 1.* Notations for the dynamics of co-orbital satellites. The satellites move around a central point mass $M$ with the same average mean motion $n_0$ and the same average orbital radius $r_0$. The equations of motion (Eqs. (1)) are written in a coordinate system with origin at $M$ and rotating with the angular velocity $n_0$. The motion is described by the coordinates $(\phi_i, \xi_i)$, $i = 1, \ldots, N$, where $\phi_i$ is the longitude of satellite $i$ with respect to an arbitrary origin, and $\xi_i = \Delta r_i / r_0$ is the relative radial excursion of that satellite with respect to the average radius $r_0$.

from which we derive:

$$f'(\phi) = \sin\phi \left[ -1 + \frac{1}{8|\sin\phi/2|^3} \right], \tag{3}$$

$$f''(\phi) = -\cos\phi - \frac{3 + \cos\phi}{16|\sin\phi/2|^3}. \tag{4}$$

The first equation in (1) represents the differential keplerian velocity of each satellite with respect to the mean orbital radius $r_0$. The second equation contains, in a derivative form, all the terms arising from the mutual interactions between the satellites. The function $f(\phi)$ in Equation (2) is the sum of the indirect and direct potentials exerted by a given satellite on the other co-orbital bodies. It is an even function of $\phi$ and its graph is plotted in Figure 2, together with the first and second derivatives $f'(\phi)$ and $f''(\phi)$.

Since $f'$ is odd, it is easy to show from (1) that $\sum_i m_i \xi_i = $ constant. Because the reference radius $r_0$ is arbitrary, it can be chosen so that $\sum_i m_i \xi_i = 0$, with no loss of generality. Thus the system (1) has the following integrals of motion:

$$\sum_i m_i \xi_i = 0, \qquad \sum_i m_i \phi_i = \text{constant}, \tag{5}$$

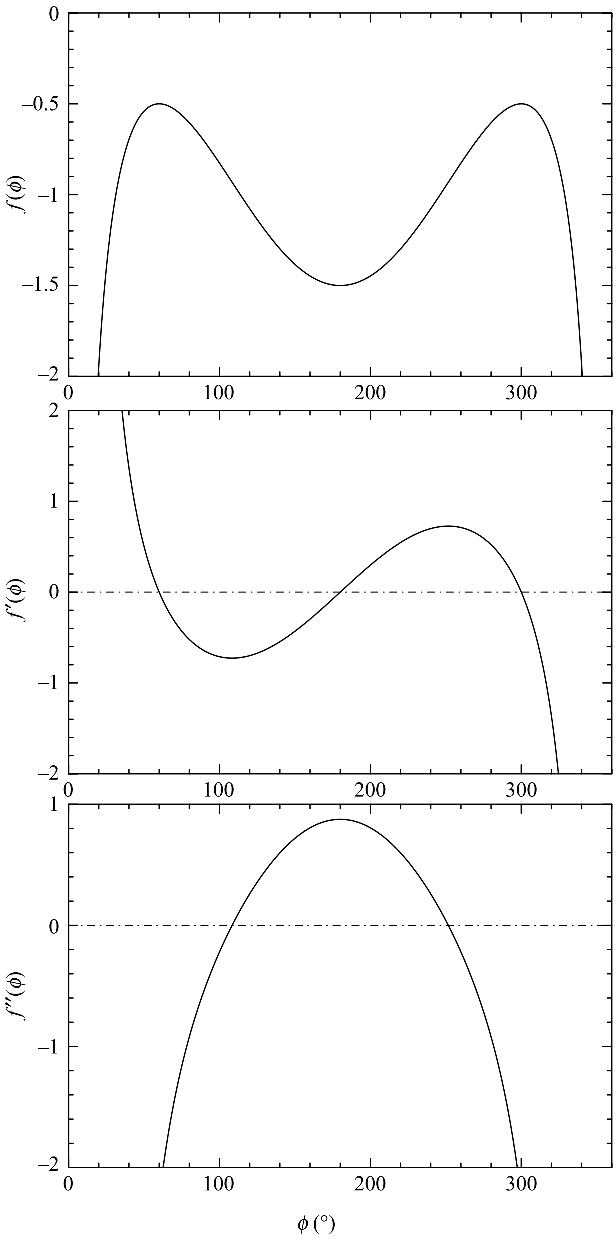


*Figure 2.* The function $f(\phi)$ (Eq. (2)), describing the potential created by a satellite on a co-orbital particle, together with its first and second derivatives $f'(\phi)$ and $f''(\phi)$ (Eqs. (3) and (4)).

which stem from the conservation of the total angular momentum. This conservation results itself from the global invariance by rotation of the whole problem considered here.

There is another integral of the system (1):

$$J = \sum_i m_i \left[ -\frac{3}{4}\xi_i^2 + \sum_{j \neq i} m_j f(\phi_i - \phi_j) \right] = \text{constant}. \tag{6}$$

This is an energy integral, called the Jacobi constant.

## 3. Equilibrium Points

### 3.1. GENERAL RESULTS

The fixed points of system (1) are given by:

$$\xi_i = 0 \tag{7}$$

and

$$\sum_{j \neq i} m_j f'(\phi_i - \phi_j) = 0 \tag{8}$$

for all $i = 1, \ldots, N$.

Equation (7) means that in a stationary configuration the $N$ co-orbital satellites have the same orbital radius.[1] Equation (8) involves the angular separations between the satellites and can be written in a matrix form. Defining $f'_{ij} \equiv f'(\phi_i - \phi_j)$ and noting that the function $f'$ is odd, we obtain from Equation (8):

$$\begin{bmatrix} 0 & f'_{12} & \cdots & \cdots & f'_{1N} \\ -f'_{12} & 0 & f'_{23} & \cdots & f'_{2N} \\ \vdots & & 0 & & \\ \vdots & & & \ddots & \\ -f'_{1N} & & & & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ \vdots \\ m_N \end{bmatrix} \equiv M_N \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ \vdots \\ m_N \end{bmatrix} = 0_{\mathbb{R}^N}. \tag{9}$$

The $N \times N$ matrix $M_N$ defined above is antisymmetric and depends only on the longitudes $\phi_i$'s, through the coefficients $f'_{ij}$.

Our aim here is to find all the angles $\phi_1, \ldots, \phi_N$ such that there are physically meaningful (i.e. positive definite) solutions $(m_1, \ldots, m_N)$ of Equation (9). It is impossible to do this analytically for an arbitrary number $N$ of satellites. However, some general properties of the solutions can be derived from the fact that $M_N$

[1] In the exact problem, this is true to zeroth order in the $m_i$'s only. Small corrections of order $m_i$ are not considered here.

is antisymmetric. We assume that for all $i \neq j$, $\phi_i \neq \phi_j$ ($f'$ is not defined on zero).

PROPOSITION 1. *Let* $(\phi_1, \ldots, \phi_N) \in [0, 360^\circ[^N$ *be such that* $\phi_i \neq \phi_j$ *for all* $i \neq j$. *If* $N$ *is odd, then the rank of* $M_N$ *is* $\mathrm{rk}(M_N) = N - k$ *where* $k$ *is an odd integer. Therefore there exists a* $k$*-parameter family of mass vectors* $(m_1, \ldots, m_N) \in I\!R^N$ *for which* $(\phi_1, \ldots, \phi_N)$ *is a stationary configuration. If* $N$ *is even, then* $\mathrm{rk}(M_N)$ *is generally* $N$, *and there is in general no family of mass vectors that leads to a stationary configuration.*

*Proof.* This results from a classical theorem of linear algebra: since $M_N$ is antisymmetric, its rank is even. Consequently, for given arbitrary angles $\phi_i$'s, the existence of non-trivial solutions (positive or not) of the linear system (9) depends on the parity of the number $N$ of satellites. □

*Remarks.* If $N$ is odd, $\mathrm{rk}(M_N) = N - k$ with $k$ odd, but $k = 1$ almost everywhere (in the space of the $\phi_i$'s, the set for which the rank of $M_N$ is $N-3$, $N-5, \ldots,$ has measure zero). Thus, given *arbitrary* non-zero angular separations between all the satellites, there is a *one-parameter* family of mass vectors for which the configuration is stationary: given for example $m_1$, there is a single non-trivial solution $(m_1, m_2, \ldots, m_N)$ of (9). Note, however, that all the masses must be positive for the solution to be physically meaningful. This will reduce the possible angular configurations $\phi_1, \ldots, \phi_N$ to a subspace of $[0, 360^\circ[^N$, as illustrated for instance in Figure 3 for the case $N = 3$.

If $N$ is even, the rank of $M_N$ is generally $N$: given an *arbitrary* angular configuration $\phi_1, \ldots, \phi_N$, there are in general *no* non-trivial solutions $(m_1, \ldots, m_N)$ which achieve the equilibrium. This is well known for the case $N = 2$, where only the triangular points $L_4$ and $L_5$ and the diametral point $L_3$ allow the satellites to be stationary with respect to each other. Thus, one must first cancel the determinant of the matrix, $\det(M_N)$, in order to obtain non-trivial solutions of (9).

Let $\mathrm{D} = \{(\phi_1, \ldots, \phi_N) \in [0, 360^\circ[^N \mid \det(M_N) = 0\}$. On this subspace, the rank of $M_N$ is $N - 2$, almost everywhere, and this time there is a *two-parameter* family of mass vectors for which the configuration is stationary. This is easily verified for $N = 2$. For instance, two satellites at 60° from each other are stationary with arbitrary choices on both $m_1$ and $m_2$.

### 3.2. APPLICATION

Let us examine some practical cases ($N \leqslant 4$), and let us search for stationary configurations with definite positive masses. For convenience, we can assume with no loss of generality that one of the satellites, for example $m_1$, is the most massive of all. In addition, the problem is invariant by rotation of the system as a whole, so we can measure all the longitudes with respect to $m_1$. More precisely, we henceforth assume that $\phi_1 = 0$, so that $\phi_i$ ($i \neq 1$) will now denote the angular separation of the $i$-th satellite with respect to $m_1$.

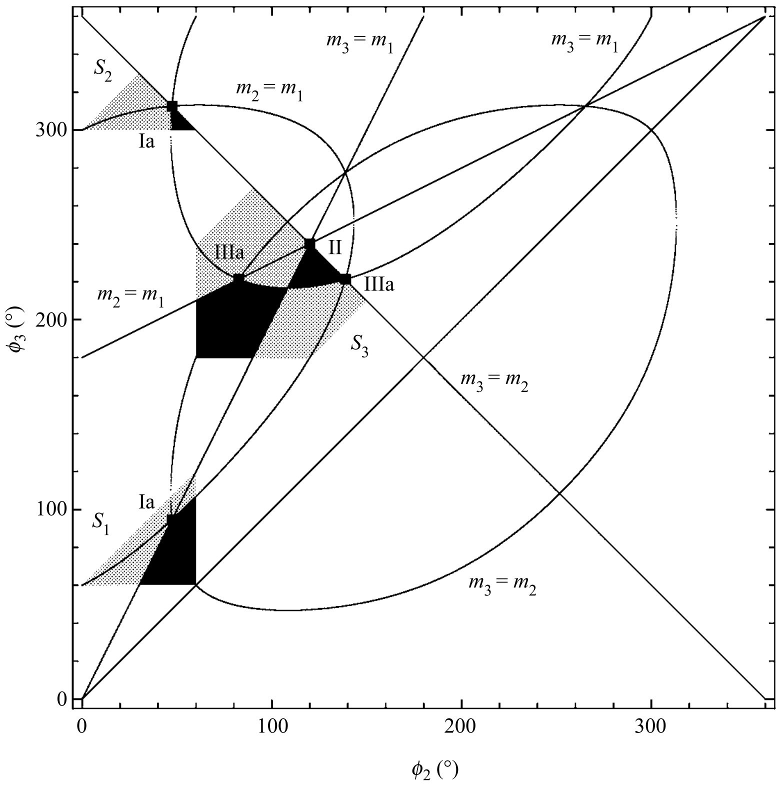


*Figure 3.* Domains of stationary configurations for three co-orbital satellites with non-zero positive masses $m_1$, $m_2$ and $m_3$. The angles $\phi_2$ and $\phi_3$ are the angular separations of $m_2$ and $m_3$ with respect to $m_1$, respectively, assumed to be at longitude $\phi_1 = 0$ with no loss of generality. As discussed in the text, it is sufficient to restrict the study to the left quadrant $0 < \phi_2 \leqslant 180°$ and $\phi_2 < \phi_3 \leqslant 360° - \phi_2$. All the possible solutions are then obtained by symmetries with respect to the two diagonals. The grey areas $S_1$, $S_2$ and $S_3$ corresponds to solutions with all three masses $m_1$, $m_2$ and $m_3$ positive, while the black areas are restricted (with no loss of generality) to cases where $m_1$ is the largest of the three masses. Each point of these areas is an equilibrium point that fixes the mass ratio between the satellites. There is an infinity of solutions that fill the three areas. The various curves and straight lines are boundaries where two masses are equal. The points Ia, II and IIIa are the stationary configurations for three identical co-orbitals, already described in SY88. Note that the particular line $\phi_3 = \phi_2$ has no physical meaning since it corresponds to two zero masses at the same point.

### 3.2.1. *The classical case of two co-orbital satellites*

With $N = 2$, the equilibrium condition yields $\xi_1 = \xi_2 = f'(\phi_2) = 0$, so that:

$$\xi_1 = \xi_2 = 0, \qquad \phi_2 = 60°, 180° \text{ or } 300°.$$

So the stationary points are the classical Lagrangian equilibrium points $L_4$, $L_3$ and $L_5$, respectively. Note that the Lagrange points $L_1$ and $L_2$ do not appear here, because we have assumed that the satellites remain far from their mutual Hill sphere.

3.2.2. *The case of three co-orbital satellites*

The equilibrium condition yields:

$$\xi_1 = \xi_2 = \xi_3 = 0, \tag{10}$$

meaning that the three co-orbitals have the same orbital radius, and

$$\begin{bmatrix} 0 & f'_{12} & f'_{13} \\ -f'_{12} & 0 & f'_{23} \\ -f'_{13} & -f'_{23} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = M_3 \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{11}$$

From the previous general result, the rank of $M_3$ is 2 (it is easy to verify that the rank cannot be zero). So, given the angular separations between the satellites, there is a one-parameter family of mass vectors that leads to a stationary configuration: given for example $m_1$, there is a single solution $(m_1, m_2, m_3)$ that satisfies Eq. (11).

Note that there are now only two independent angular variables, $\phi_2$ and $\phi_3$, from the choice $\phi_1 = 0$.

We only search the non-zero positive solutions for $m_1$, $m_2$ and $m_3$. Solving the linear system (11) then yields:

$$\frac{m_2}{m_1} = -\frac{f'_{13}}{f'_{23}} > 0, \qquad \frac{m_3}{m_1} = \frac{f'_{12}}{f'_{23}} > 0. \tag{12}$$

From this system, one can deduce immediately that if $(\phi_2, \phi_3)$ is solution, then $(\phi_3, \phi_2)$, $(360° - \phi_2, 360° - \phi_3)$ and $(360° - \phi_3, 360° - \phi_2)$ are solutions too. Thus, without loss of generality, we can restrict the search for the stationary configurations to the following subspace:

$$0 < \phi_2 < \phi_3 < 360° - \phi_2.$$

This restricts our search to the left quadrant of the total space $[0, 360°[^2$ available to $(\phi_2, \phi_3)$, see Figure 3. Geometrically, this means that the three satellites $m_1$, $m_2$ and $m_3$ are always ordered in the direct trigonometric sense, that $\phi_2$ is always between 0° and 180°, and that the satellite closest to $m_1$ is always $m_2$.

In this subspace, one can find analytically from the relations (12) all the angular separations between the satellites that correspond to a stationary configuration with all masses definite positive, see the grey shaded regions in Figure 3. There are three areas that contain an infinity of equilibrium points, each associated with a mass ratio $m_3/m_1$ and $m_2/m_1$ given by (12). These three areas are:

$$\begin{aligned} &S_1\colon\quad 0 < \phi_2 < 60°, \quad 60° < \phi_3 < \phi_2 + 60°, \\ &S_2\colon\quad 0 < \phi_2 < 60°, \quad 300° < \phi_3 < \min(\phi_2 + 300°, 360° - \phi_2), \\ &S_3\colon\quad 60° < \phi_2 < 150°, \\ &\quad \max(180°, \phi_2 + 60°) < \phi_3 < \min(\phi_2 + 180°, 360° - \phi_2). \end{aligned}$$

As stated above, $m_1$ can moreover be assumed to be the most massive satellite. This restricts the equilibrium point $(\phi_2, \phi_3)$ to be in one of the black regions of Figure 3, where all the mass ratios $m_2/m_1$ and $m_3/m_1$ in the $]0, 1]$ interval are represented.

Since the assumption that $m_1$ is the most massive satellite can be made without loss of generality, it follows that the grey regions in Figure 3 can be reduced to the black regions, through an appropriate permutation in the indexes $i = 1, 2, 3$. The black regions thus concentrate all the possible co-orbital stationary configurations for $N = 3$ satellites, to within the approximation $m_i \ll 1$ for all $i = 1, \ldots, N$.

The boundaries between the grey and black regions are straight lines or oval-shaped curves where $m_2$ or $m_3$ is equal to $m_1$. The line $\phi_3 = \phi_2$ must be excluded because it corresponds to two zero masses $m_2$ and $m_3$ at the same point. The three lines $\phi_3 = \phi_2 + 60°$, $\phi_3 = \phi_2 + 180°$ and $\phi_3 = \phi_2 + 300°$ (not plotted in Figure 3) must be excluded as well because they correspond to ratios $m_2/m_1$ or $m_3/m_1$ going to infinity, with $m_2$ and $m_3$ separated by 60°, 180° or 300°, thus meaning that we are back to the problem of $N = 2$ co-orbital satellites only. The same is true for $\phi_2 = 60°$ (corresponding to $m_3 = 0$) and for $\phi_3 = 60°$ or 180° or 300° (corresponding to $m_2 = 0$).

The equations of the oval-shaped curves of Figure 3 are obtained numerically from Equation (12). The oval-shaped curves $m_2 = m_1$, $m_2 = m_3$ and $m_3 = m_1$ are given by $f'(\phi_3) + f'(\phi_3 - \phi_2) = 0$, $f'(\phi_2) + f'(\phi_3) = 0$ and $f'(\phi_2) + f'(\phi_3 - \phi_2) = 0$, respectively. All the lines or curves where any two of the masses are equal have been labeled accordingly in Figure 3.

There are three stationary configurations with equal masses. These configurations are already described in SY88 and referred to as types Ia, II and IIIa. They appear in Figure 3 with the same notation. One is when all three satellites are equally spaced by 120°, see label II in Figure 3. A second configuration is when the angular separation between two adjacent satellites is 47°.361. This configuration appears twice in the quadrant considered in Figure 3 (labels Ia). The third possible stationary configuration with equal masses occurs when two satellites are 138°.765 away from the third one (and thus at 82°.470 from each other). This configuration also appears twice in Figure 3, see labels IIIa.

#### 3.2.3. *The case of four co-orbital satellites*

The stationarity now requires:

$$\xi_1 = \xi_2 = \xi_3 = \xi_4 = 0, \tag{13}$$

meaning that the satellites have the same orbital radius, and

$$\begin{bmatrix} 0 & f'_{12} & f'_{13} & f'_{14} \\ -f'_{12} & 0 & f'_{23} & f'_{24} \\ -f'_{13} & -f'_{23} & 0 & f'_{34} \\ -f'_{14} & -f'_{24} & -f'_{34} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{14}$$

The $4 \times 4$ matrix is antisymmetric, the rank is even. Therefore, in order to have non-trivial solutions, one must first cancel the matrix determinant, that is:

$$f'_{14}f'_{23} + f'_{12}f'_{34} - f'_{13}f'_{24} = 0. \tag{15}$$

The actual determinant of the matrix is the square of the left-hand side of the equation above, commonly called the Pfaffian of the antisymmetric matrix. If (15) is satisfied, then the matrix rank is 2 (again, it is easy to verify that it cannot be zero). Therefore there is a two-parameter family of mass vectors for which the configuration is stationary: given for example $m_1$ and $m_2$, one deduce $m_3$ and $m_4$, solving Equation (14). In fact, in order to have non-zero positive solutions, the resolution of the linear system yields:

$$\operatorname{sgn}(f'_{34}) \times \left[ f'_{14} + f'_{24}\frac{m_2}{m_1} \right] < 0, \qquad \operatorname{sgn}(f'_{34}) \times \left[ f'_{13} + f'_{23}\frac{m_2}{m_1} \right] > 0, \tag{16}$$

where sgn denotes the sign.

Thus, for angles which are solutions of (15), and for a given $m_1$, there is an infinity of solutions that depend on the choice of $m_2$ that satisfies the relations (16). The same symmetry arguments as those quoted previously for three satellites enable us to search the stationary configurations in the following subspace: $\{(\phi_2, \phi_3) \mid 0 < \phi_2 < \phi_3 \leqslant 180°, \phi_3 < \phi_4 < 360°\}$.

The Equations (16) cannot be solved analytically for any arbitrary ratio $m_2/m_1$. Therefore, for $N \geqslant 4$, the possible stationary configurations can only be derived using numerical schemes, see Section 5.

## 4. Linear Stability of the Stationary Configurations

### 4.1. General Results

Given a stationary configuration $\phi_s = (\phi_{1s}, \phi_{2s}, \ldots, \phi_{Ns})$ and $\xi_s = (\xi_{1s}, \xi_{2s}, \ldots, \xi_{Ns})$ which is a solution of (7) and (8), we classically expand the equations of motions (1) to first order in $\Delta\phi$ and $\Delta\xi$, where:

$$\Delta\phi \equiv \begin{bmatrix} \phi_1 - \phi_{1s} \\ \phi_2 - \phi_{2s} \\ \vdots \\ \phi_N - \phi_{Ns} \end{bmatrix} \quad \text{and} \quad \Delta\xi \equiv \begin{bmatrix} \xi_1 - \xi_{1s} \\ \xi_2 - \xi_{2s} \\ \vdots \\ \xi_N - \xi_{Ns} \end{bmatrix}.$$

This yields:

$$\begin{bmatrix} \Delta\dot{\phi} \\ \Delta\dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2}I_N \\ -2A & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi \\ \Delta\xi \end{bmatrix} \equiv L_N \begin{bmatrix} \Delta\phi \\ \Delta\xi \end{bmatrix}, \tag{17}$$

where:

$$A=\begin{bmatrix}\sum_{j\neq 1} m_j f''_{1j} & -m_2 f''_{12} & \cdots & -m_N f''_{1N}\\ -m_1 f''_{12} & \sum_{j\neq 2} m_j f''_{2j} & \cdots & -m_N f''_{2N}\\ \vdots & & \ddots & \vdots\\ -m_1 f''_{1N} & -m_2 f''_{2N} & & \sum_{j\neq N} m_j f''_{Nj}\end{bmatrix}, \tag{18}$$

$I_N$ is the $N \times N$ identity matrix, and $f''_{ij} \equiv f''(\phi_i - \phi_j)$. Note that when writing the expression of $A$, we have used the fact that $f''$ is an even function of $\phi$, that is, $f''_{ij} = f''_{ji}$.

The eigenvalues of the matrix $L_N$ defined in (17) are the eigenfrequencies of the system near the stationary configuration $(\xi_s, \phi_s)$.

PROPOSITION 2. *Within the assumption that* $m_i \geqslant 0$ *for all* $i = 1, \ldots, N$, *the* $2N$ *eigenvalues* $\lambda_1, \lambda_1, \ldots, \lambda_{2N}$ *of the* $2N \times 2N$ *matrix* $L_N$ *are either purely real or purely imaginary. Thus, a stationary configuration is linearly stable if all the eigenvalues are purely imaginary.*

*Proof.* Consider the eigenvector $(\Delta\phi_\lambda, \Delta\xi_\lambda)$ associated with an eigenvalue $\lambda$ of $L_N$. From the structure of Equation (17), it is easy to see that $(\Delta\phi_\lambda, -\Delta\xi_\lambda)$ is also an eigenvector of $L_N$ with eigenvalue $-\lambda$. Furthermore, it is easy to show that

$$A \cdot \Delta\phi_\lambda = \tfrac{1}{3}\lambda^2 \Delta\phi_\lambda.$$

Conversely, if $A \cdot \Delta\phi = \alpha\Delta\phi$ ($\alpha$ complex), then one can see that $(\Delta\phi, \Delta\xi = \mp\sqrt{4\alpha/3}\Delta\phi)$ are eigenvectors of $L_N$ associated with the eigenvalues $\pm\sqrt{3\alpha}$.

Thus, searching for the $2N$ eigenvalues $\lambda$'s of $L_N$ is equivalent to searching for the $N$ eigenvalues $\alpha$'s of $A$ and then taking

$$\lambda = \pm\sqrt{3\alpha}. \tag{19}$$

Although $A$ is *not* symmetric, its eigenvalues are real. This can be shown by considering the symmetric matrix $S$ constructed by multiplying the first row of $A$ by $m_1$, the second row by $m_2$, etc.:

$$S=\begin{bmatrix}\sum_{j\neq 1} m_1 m_j f''_{1j} & -m_1 m_2 f''_{12} & \cdots & -m_1 m_N f''_{1N}\\ -m_1 m_2 f''_{12} & \sum_{j\neq 2} m_2 m_j f''_{2j} & \cdots & -m_2 m_N f''_{2N}\\ \vdots & & \ddots & \vdots\\ -m_1 m_N f''_{1N} & -m_2 m_N f''_{2N} & \cdots & \sum_{j\neq N} m_j m_N f''_{1N}\end{bmatrix}, \tag{20}$$

If $X_\alpha = (X_{1\alpha}, X_{2\alpha}, \ldots, X_{N\alpha})$ is an eigenvector of $A$ with eigenvalue $\alpha$, then

$$S \cdot X_\alpha = \alpha Y_\alpha,$$

where $Y_\alpha = (m_1 X_{1\alpha}, m_2 X_{2\alpha}, \dots, m_N X_{N\alpha})$. Since $S$ is real and symmetric, we have $S = \bar{S} = S^{\mathrm{T}}$, where the bar denotes the complex conjugate, and the superscript T denotes the transpose operator. From $S \cdot X_\alpha = \alpha Y_\alpha$, we get successively:

$$X_\alpha^{\mathrm{T}} S^{\mathrm{T}} = \alpha Y_\alpha^{\mathrm{T}}, \qquad \bar{S} \cdot \bar{X}_\alpha = S \cdot \bar{X}_\alpha = S^{\mathrm{T}} \cdot \bar{X}_\alpha = \bar{\alpha} \bar{Y}_\alpha.$$

Multiplying both sides of the last equation by $X_\alpha^{\mathrm{T}}$, we obtain:

$$\alpha Y_\alpha^{\mathrm{T}} \bar{X}_\alpha = \bar{\alpha} X_\alpha^{\mathrm{T}} \bar{Y}_\alpha.$$

Since the $m_i$ are non-negative, $X_\alpha^{\mathrm{T}} \bar{Y}_\alpha = Y_\alpha^{\mathrm{T}} \bar{X}_\alpha = m_1 |X_{1\alpha}|^2 + m_2 |X_{2\alpha}|^2 + \cdots + m_N |X_{N\alpha}|^2 \neq 0$, so that $\bar{\alpha} = \alpha$, showing that $\alpha$ is real. Therefore, $\lambda = \pm\sqrt{3\alpha}$ is either purely real or purely imaginary. □

Finally, we note that $\det(A) = 0$ since the columns of the matrix are not linearly independent: adding all the columns yields the $N$-dimensional zero vector. Thus, at least one of the eigenvalues of $A$ (and at least two eigenvalues of $L_N$) is zero. Note that this is for the same reason that we have $\sum_i m_i \xi_i = 0$. This expresses the conservation of angular momentum commented after Equation (5). This conservation eventually stems from the invariance by rotation of the problem, as already noted by SY88. This double zero eigenvalue of $L_N$ actually disappears if one lowers the dimension of the system by the change of variables where one of the satellites, for example, $m_1$, is considered as the origin of coordinates ($u_i = \xi_i - \xi_1$, $v_i = \phi_i - \phi_1$).

In summary, the characteristic polynomial $P(x)$ of the $2N \times 2N$ matrix $L_N$ is an even polynomial of the form $P(x) = x^2 Q(x^2)$, where $Q(y)$ is a polynomial of degree $N-1$. As seen above, $yQ(y)$ is proportional to the characteristic polynomial of $A$, and all the roots $\alpha$ of $Q$ are real. Consequently, from (19), the eigenvalues $\lambda$ of $L_N$ are either purely real or purely imaginary (or zero).

The condition of linear stability is thus that all the non-trivial eigenvalues $\lambda$'s [that is, the roots of $Q(x^2) = 0$] are purely imaginary, that is, that all the roots of $Q(y) = 0$ are negative. This can easily be verified in the case of the lagrangian stationary points with $N = 2$ satellites, for which the eigenfrequencies associated with $L_4$ or $L_5$ are purely imaginary, $\lambda = \pm \mathrm{j}\sqrt{-3(m_1 + m_2) f''(\phi_s)} = \pm \mathrm{j}\sqrt{27(m_1 + m_2)/4}$, where $\mathrm{j} = \sqrt{-1}$ and $\phi_s = 60°$ or $300°$.

Notice in passing that for some specific stationary configurations, some of the roots of $Q(y) = 0$ may *also* be zero, independently of the zero values associated with the rotational invariance discussed above, see an example below for $N = 3$ satellites.

### 4.2. APPLICATION TO THE CASE OF THREE CO-ORBITAL SATELLITES

Besides the double eigenfrequency $\lambda = 0$, the other four non-trivial eigenfrequencies are given by:

$$\begin{aligned} Q(x^2) \;=\; & x^4 - [(m_1 + m_2) f''_{12} + (m_2 + m_3) f''_{23} + (m_3 + m_1) f''_{31}] x^2 + \\ & (m_1 + m_2 + m_3)[m_1 f''_{12} f''_{31} + m_2 f''_{23} f''_{12} + m_3 f''_{31} f''_{23}] = 0, \end{aligned}$$

where we can use $f''_{ij} = f''_{ji}$. The two real roots of $Q(y) = 0$ have the following sum and product:

$$s = (m_1 + m_2) f''_{12} + (m_2 + m_3) f''_{23} + (m_3 + m_1) f''_{31},$$
$$p = (m_1 + m_2 + m_3)[m_1 f''_{12} f''_{31} + m_2 f''_{23} f''_{12} + m_3 f''_{31} f''_{23}], \tag{21}$$

so that the linear stability (roots of $Q(x^2) = 0$ purely imaginary) of a given stationary configuration requires:

$$s < 0, \qquad p > 0. \tag{22}$$

The domains in the $(\phi_2, \phi_3)$ plane where both conditions are fulfilled are plotted in Figure 4 as light grey areas. Practically, the only linearly stable stationary configurations with $N = 3$ satellites are those enclosed in the two triangular areas $S_1$ and

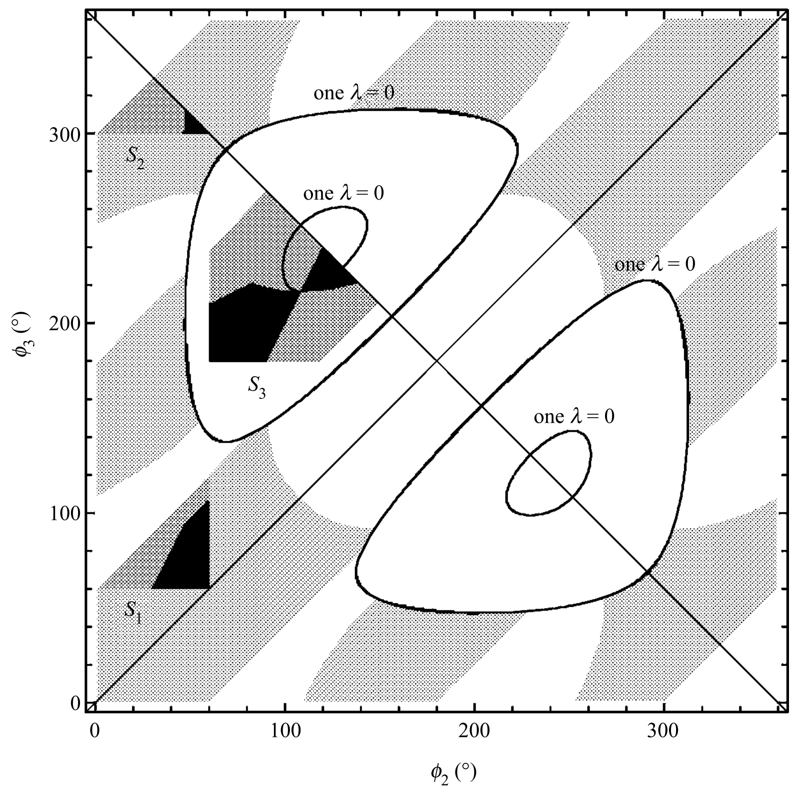


*Figure 4.* Stability of the stationary configurations for $N = 3$ satellites. The black and dark grey areas have the same meaning as in Figure 3, namely the only physically meaningful stationary configurations (all three masses positive), restricted to the left quadrant for clarity. The light grey areas correspond to the domains of linear stability, that is, where the eigenvalues of $L_N$ are all purely imaginary. Thus only the stationary configurations in the two triangular areas $S_1$ and $S_2$ are linearly stable. The four closed curves are the sets of points where the product $p$ (Eq. (21)) is zero, that is, where one of the non-trivial eigenfrequencies $\lambda$ is zero, see text for details.

$S_2$ defined before and shown in Figures 3 and 4. They correspond to configurations where the less two massive satellites $m_2$ and $m_3$ are either positioned on each side of $m_1$, near the $L_4$ and $L_5$ points of $m_1$, or are grouped together near one of these two points, see examples in Figure 5.

The four closed curves in Figure 4 indicate the points where one of the roots of $Q(x^2) = 0$ is zero, that is, where

$$p = (m_1 + m_2 + m_3)[m_1 f''_{12} f''_{31} + m_2 f''_{23} f''_{12} + m_3 f''_{31} f''_{23}] = 0.$$

At those points, one of the non-trivial eigenfrequencies of $L_N$ is zero, indicating that one of the satellites is at equilibrium at a singular point which suffers a transition between elliptic and hyperbolic topologies. At that point, the equilibrium is neutral, hence the zero eigenfrequency.

## 5. Numerical Search for Stable Stationary Configurations

Because the analytical resolution of Equation (9) is untractable for $N \geqslant 4$, a simple numerical scheme is useful for finding the possible linearly stable stationary configurations, given the masses $m_1, m_2, \ldots, m_N$.

The idea is to add in the equations of motion a non-conservative term so that the satellites will converge towards the linearly stable stationary configurations. Specifically, let us consider the perturbed equations of motions:

$$\dot{\phi}_i = -\frac{3}{2}\xi_i, \qquad \dot{\xi}_i = -2\sum_{j\neq i} m_j f'(\phi_i - \phi_j) - \nu\xi_i, \tag{23}$$

where $\nu$ is a positive real number. It is straightforward to see that near a (conservative) stationary point, the linearized equations of motions are:

$$\begin{bmatrix} \Delta\dot{\phi} \\ \Delta\dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2}I_N \\ -2A & -\nu I_N \end{bmatrix} \begin{bmatrix} \Delta\phi \\ \Delta\xi \end{bmatrix} \equiv L_{N,\nu} \begin{bmatrix} \Delta\phi \\ \Delta\xi \end{bmatrix}, \tag{24}$$

and that the eigenvalues $\lambda$'s of the new matrix $L_{N,\nu}$ are given by:

$$\lambda^2 + \nu\lambda = 3\alpha, \tag{25}$$

where the $\alpha$'s are the real eigenvalues of $A$. If $\alpha$ is positive, then the two solutions of (25) are real, with one of them positive. This shows that a linearly unstable configuration in the conservative case remains linearly unstable when introducing the non-conservative term $-\nu\xi$ in Equation (23).

Conversely, if $\alpha$ is negative, then the solutions of (25) are real or complex, but the real parts of these solutions are negative anyway, because $\nu > 0$. Thus a linearly stable configuration in the conservative case will locally be an attractor for the solutions of Equation (23).

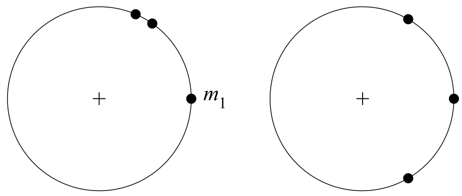


*Figure 5.* Stable stationary configurations for $N = 3$ co-orbital satellites with $m_2 = m_3 = 10^{-2}m_1$. The satellite $m_1$ is at $\phi_1 = 0$. To within the possible symmetries discussed in the text, there are exactly two linearly stable solutions in this case. From left to right, the angular separations of satellites 2 and 3 with respect to satellite 1 are: $(\phi_2, \phi_3) = (54°.84, 66°.75), (59°.82, 300°.18)$.

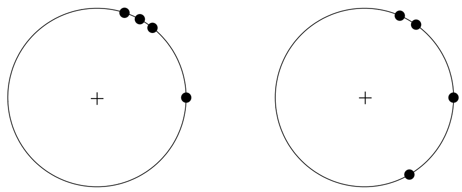

*Figure 6.* The same as for the previous figure, but with $N = 4$ co-orbital satellites with $m_2 = m_3 = m_4 = 10^{-2}m_1$. To within the possible symmetries, there are two linearly stable solutions in this case. From left to right, the angular separations of satellites 2–4 with respect to satellite 1 are: $(\phi_2, \phi_3, \phi_4) = (51°.46, 61°.29, 71°.91), (54°.67, 66°.56, 300°.35)$.

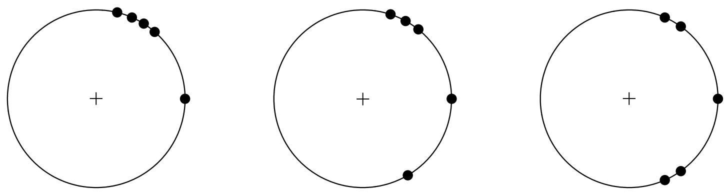

*Figure 7.* The same as for the previous figure, but with $N = 5$ co-orbital satellites with $m_2 = m_3 = m_4 = m_5 = 10^{-2}m_1$. To within the possible symmetries, there are three linearly stable solutions in this case. From left to right, the angular separations of satellites 2–5 with respect to satellite 1 are: $(\phi_2, \phi_3, \phi_4, \phi_5) = (48°.92, 57°.64, 66°.29, 76°.29)$, $(51°.31, 61°.11, 71°.71, 300°.52)$, $(54°.52, 66°.37, 293°.63, 305°.48)$.

Physically, the introduction of the term $-\nu\xi$ in Equation (23) brings energy to the system in the rotating frame of the co-orbital satellites. More precisely, this term increases the Jacobi constant (6) since:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = -\frac{3}{2}\sum_i m_i \xi_i \dot{\xi}_i \ldots = \frac{3}{2}\nu \sum_i m_i \xi_i^2 \ldots > 0. \tag{26}$$

This forces the solution to converge towards a local maximum of $J$ (with respect to small variations of $\xi_i$ and $\phi_i$). These local maxima actually correspond to the linearly stable configurations, a property demonstrated by Moeckel (1994).

Integrating numerically Equation (23) with a suitable value of $\nu$, and using random initial coordinates with given mass ratios $0 < m_i/m_1 \leqslant 1$ $(i \neq 1)$, we can find the stable stationary configurations.

Exploring various positive values of $m_1, m_2, \ldots, m_N$ then provides the domains of stable stationary points. We have checked that for the case of $N = 3$ satellites, we retrieve the areas found analytically and shown in Figure 3. We have also noticed that for $N$ small but $N \geqslant 4$, the linearly stable stationary points correspond to configurations where the co-orbitals are either positioned near the $L_4$ and $L_5$ points of the most massive satellite, or are grouped near one of these two points.

For sake of illustration, we give here some typical cases of stable stationary configurations. In all Figures 5–7, the satellite $m_1$ is the most massive and is located at $\phi_1 = 0$ (the right-most position).

## 6. Conclusions

We have derived general results concerning the existence of stationary configurations for $N$ co-orbital satellites with non-zero and small (but otherwise arbitrary) masses revolving around a massive primary.

The existence of stationary configurations depends on the parity of $N$. If $N$ is odd, then for any arbitrary choice of longitudes for each satellite, $\phi_1, \phi_2, \ldots, \phi_N$, with $\phi_i \neq \phi_j$ when $i \neq j$, there always exists a set of masses (positive or negative) $m_1, m_2, \ldots, m_N$ for which the stationarity is achieved. However, because we require that $m_i > 0$ for all $i$, the physically acceptable solutions actually correspond to sub-domains of $(\phi_1, \phi_2, \ldots, \phi_N)$. Example of such domains are given in the case of $N = 3$ satellites, see Figure 3. In general the solution $m_1, m_2, \ldots, m_N$ is a one-parameter family scaled, for instance, by $m_1$. It can be a three-parameter, five-parameter family, but only on a space of measure zero.

If $N$ is even, then for a given angular configuration $\phi_1, \phi_2, \ldots, \phi_N$, there is in general *no* non-trivial solution $m_1, m_2, \ldots, m_N$ for which the stationarity is achieved. When the $\phi_1, \phi_2, \ldots, \phi_N$ configuration permits the existence of a non-trivial solution (which happens when $\det(M_N) = 0$), then there is in general a two-parameter family of masses $m_1, m_2, \ldots, m_N$ which achieves the stationarity.

The linear stability of the stationary configurations has been investigated. The eigenfrequencies of the system near stationarity are either purely real or purely imaginary, or zero. These eigenfrequencies are the square roots of all the eigenvalues (which are real) of the matrix $A$ defined in (18).

Our results apply to the limiting case where $m_i = \epsilon\mu_i$, $i = 1, \ldots, N$, where $\mu_i \in I\!R^+$ and $\epsilon \longrightarrow 0$. The more general problem where $\epsilon$ does not tend to zero is non-trivial and beyond the scope of this paper. Indeed, not all of the stationary configurations with $\epsilon \longrightarrow 0$ are limits of equilibria with $\epsilon$ not tending to zero. A family of relative equilibria $x^\epsilon$ converging, as $\epsilon \longrightarrow 0$, to a stationary configuration $x$ of the $(1 + N)$ body problem exists only if $x$ is a *non-degenerate* local minimum of the potential $V = \sum_{i<j} \mu_i \mu_j [1 - f(\phi_i - \phi_j)]$ (Hessian of $V$ positive semi-definite with nullity equal to one). Moreover, $x^\epsilon$ is linearly stable for $\epsilon$ sufficiently small if and only if $x$ is a local maximum of $f$. These questions are discussed in Hall (1987) and Moeckel (1994).

We have not treated the global (non-linear) stability of the stationary configurations. This is well outside the scope of this paper. Note however that, within the frame of the restricted circular 3-body problem, the problem has been completely answered in the planar case, with the essential aid of KAM theory. Some progress has also been made in the spatial case: it has been shown for instance that $L_4$ and $L_5$ are *Nekhoroshev-stable* (namely stable over exponentially long times) for all but a few values of the reduced mass up to the Routh critical value $\mu \sim 0.0385209$ (Benettin et al., 1998).

The case $N = 3$ is treated completely in the frame of our approximations. Figures 3 and 4 give all the possible solutions with positive masses, and their linear stability.

Finally, we propose a numerical scheme (the integration of Eq. (23)) to find rapidly the possible linearly stable stationary configurations for a given set of masses $m_1, m_2, \ldots, m_N$. Conversely, if an angular configuration $\phi_1, \phi_2, \ldots, \phi_N$ is given, the inversion of Equation (9) provides the corresponding solution $m_1, m_2, \ldots, m_N$. This is an easy task for modest values of $N$, using classical numerical schemes to search for the eigenvectors of the matrix $M_N$.

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