Vlasov equilibria with density and temperature inhomogeneities

F. Pegoraro, C. Montagna

Meudon 14-09-2007

Dipartimento di Fisica Università di Pisa pegoraro@df.unipi.it

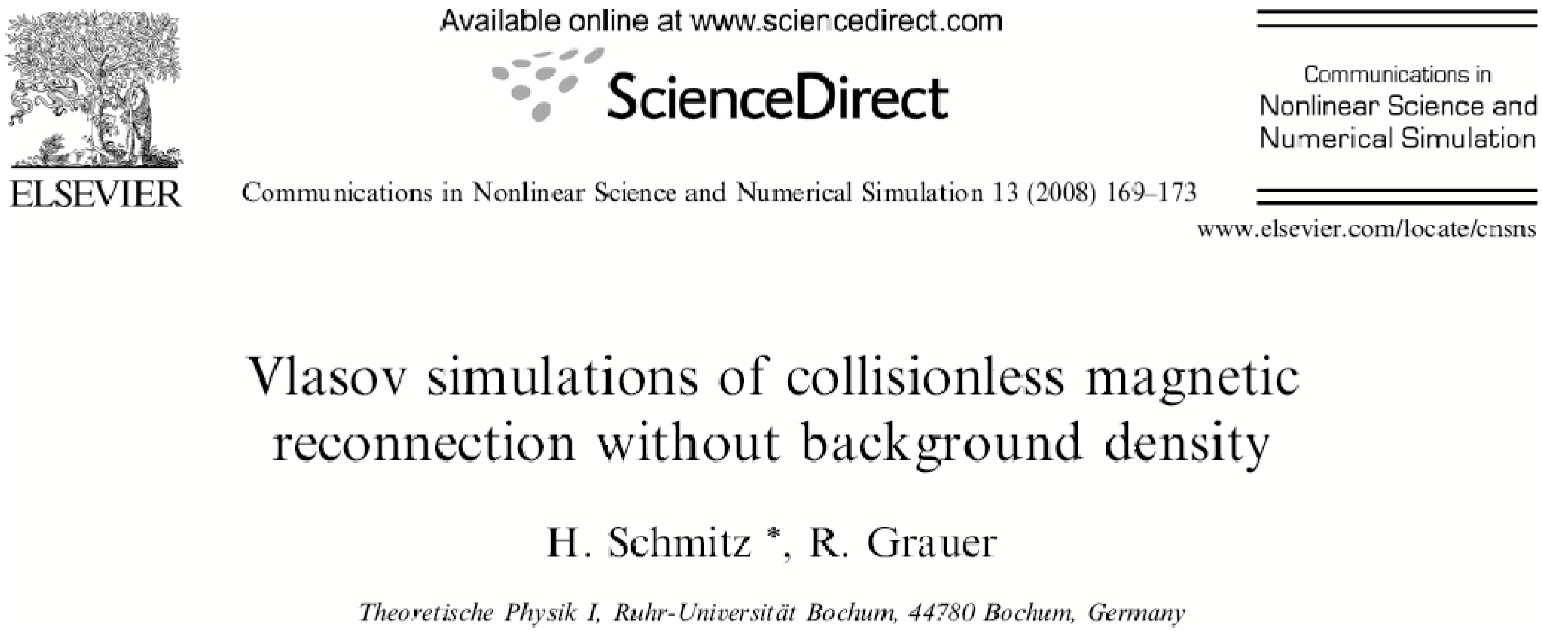
International workshop on **SPACE PLASMAS AND ASTROPHYSICS** in honor of **ANDRE' MANGENEY**

VLASOV EQUILIBRIA WITH DENSITY AND TEMPERATURE INHOMOGENEITIES

Francesco Pegoraro, Chiara Montagna, Pisa Italy

Dipartimento di Fisica Università di Pisa pegoraro@df.unipi.it

It seems that even in future years studies of collisionless magnetic reconnection, in particular numerical studies, will require the knowledge of appropriate Vlasov equilibrium configurations as starting points: see date in Fig.(1)


Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Communications in Nonlinear Science and Numerical Simulation 13 (2008) 169–173

Communications in Nonlinear Science and Numerical Simulation

www.elsevier.com/locate/cnsns

Vlasov simulations of collisionless magnetic reconnection without background density

H. Schmitz *, R. Grauer

Theoretische Physik I, Ruhr-Universität Bochum, 44780 Bochum, Germany

Figure 1: Paper from **2008**

In fact, stationary solutions (equilibria) of the Vlasov-Maxwell system based on Jeans' theorem[1] provide a convenient starting point in the investigation of the nonlinear dynamics of electromagnetic plasmas in collisionless regimes[2] and of stellar systems such as galaxies (with the gravitational potential replacing the electromagnetic potentials)[3].

[1] J.H. Jeans, Mon. Not. R. Astr. Soc. **76**, 71 (1915);
D. Lynden-Bell, Mon. Not. R. Astr. Soc. **124**, 1 (1962).
[2] E.G. Harris, Il Nuovo Cimento **23**, 115 (1962).
S.M. Mahajan, Phys. Fluids B **1**, 43 (1989);
S. M. Mahajan, W.-Q. Li, Phys. Fluids B **1**, 2345 (1989);
N. Attico, F. Pegoraro, Phys. Plasmas **6**, 767 (1999);
N.A. Bobrova, S.V. Bulanov, J.I. Sakai, D. Sugiyama, Phys. Plasmas **8**, 759 (2001);
F. Mottez, Phys. Plasmas **10**, 2501 (2003) ;
F. Ceccherini, C. Montagna, F. Pegoraro, G. Cicogna Phys. Plasmas, **12**, 052506 (2005);
C. Montagna, F. Pegoraro, CNSNS, **13**, 147, (2008);
C. Montagna, F. Pegoraro, Phys. Plasmas, **14**, 2103, (2007) .
[3] G. Bertin, M. Stiavelli, Rep. Prog. Phys. **56** 493 (1993) and references therein;
J. Perez, CNSNS, **13**, 153 (2008).

In particular, isothermal equilibria with a nonuniform density are frequently considered because, among other reasons, they may be expected to be more resilient to the long term dissipative effects of particle collisions.
In addition, they lead to physical models that are relatively simple to solve algebraically, although such models are often affected by divergences (as e.g., in isothermal stellar systems) or by unphysical boundary conditions.

On the other hand plasma equilibria with nonuniform temperature distributions are of great interest as temperature gradients are known to affect the dynamics of magnetically confined plasmas, giving rise to new instabilities[4] or modifying important plasma processes such as magnetic reconnection[5].

[4]B. Coppi, M.N. Rosenbluth, R.Z Sagdeev, ICTP Trieste Report IC/66/24 (1966);
B. Coppi, F. Pegoraro, Nuclear Fusion, **17**, 969 (1977).
[5]B. Coppi, J.W.-K. Mark, L. Sugiyama, G. Bertin, Phys. Rev. Lett., **42**, 1058 (1979).

The Harris pinch

The well known Harris pinch equilibrium describes a purely magnetic (i.e., fully neutral), isothermal one-dimensional stationary plasma configuration embedded in a magnetic field of the form

$$\mathbf{B}(x) = B_{\mathrm{y}}(x)\mathbf{e}_{\mathrm{y}} + B_{\mathrm{z}}\mathbf{e}_{\mathrm{z}},$$

with B_{z} constant, possibly zero.
The particle kinetic energy $\epsilon_j \equiv m_j \mathbf{v}^2/2$ and the z component of the canonical momentum $p_{j\mathrm{z}} = m_j v_{\mathrm{z}} + Z_j e A_{\mathrm{z}}/c$ are integrals of the particle motion. $j = e, i$ particle species Z_j, m_j particle charge number and mass.
Any distribution function of the form $f_j(x, \mathbf{v}) = F_j(\epsilon_j, p_{j\mathrm{z}})$ *that satisfies the appropriate positivity and integrability conditions is a stationary solution of Vlasov's equation.*

The set of the Vlasov-Maxwell equations is then closed by calculating the current density along z and by solving Ampère's equation

$$\nabla^2 A_{\rm z}(x) = -4\pi \sum_j \ [Z_j e \int d^3v \ v_{\rm z} \ F_j(\epsilon_j, p_{\rm z}) \ ]/c,$$

for the unknown vector potential, after imposing that the particle densities

$$n_j = \int d^3v \ F_j(\mathbf{v}^2, p_{\rm z})$$

be equal, i.e., that the configuration is charge neutral. This allows us to find the spatial dependence of the magnetic field selfconsistently.

The Harris solution[6] is obtained by choosing two isothermal distribution functions of the form

$$F_j(\epsilon_j, p_{j\mathrm{z}}) = \frac{n_{0j}}{(2\pi T_j/m_j)^{3/2}} \; \exp\left[(-\epsilon_j + u_j^* p_{j\mathrm{z}} - m_j u_j^{*2}/2)/T_j\right] \,, \tag{1}$$

where n_{0j} is a reference density and u_j^* is the stream velocity parameter.
The neutrality condition requires that the combinations $|Z_j| n_{0j}$ and $Z_j u_j^*/T_j$ be equal for electron and ions.
For the (diamagnetic) current density we obtain

$$J_\mathrm{z}(A_\mathrm{z}(x,y)) = \sum_j \; Z_j e u_j^* n_{0j} \exp\left[(u_j^* Z_j e A_\mathrm{z}/c)/T_j\right] \,. \tag{2}$$

[6] E.G. Harris, Il Nuovo Cimento **23**, 115 (1962).

The distributions (1) can be written as the product of shifted Maxwellians (centered at u_j^*) times a common density distribution. The particle and the current densities have the same space dependence and the sign of the current density is the same all over the configuration[7].
By solving Ampère's equation we obtain

$$A_{\mathrm{z}}(x) = -\ln\left[\cosh\left(x\right)\right], \quad B_{\mathrm{y}}(x) = -\tanh\left(x\right), \quad n(x) = 1/\cosh^2(x), \tag{3}$$

where we have normalized the vector potential on $2cT_i/(Z_i e u_i^*)$, n on the maximum density and x on $cT_i/[Z_i e u_i^* (2\pi Z_i n_{0i}(T_e + T_i))^{1/2}]$.

[7]An additional uniform Maxwellian distribution can be added without changing the current distribution.

2-D Harris-Liouville equilibria

The Harris pinch equilibrium is a special case of a class of two dimensional isothermal configurations that are obtained simply by setting $A_{\rm z}(x,y)$ into Eq.(1). Ampère's equation takes the form of the nonlinear Liouville equation in 2-D

$$\nabla^2 A_{\rm z}(x,y) = -\exp\left[2A_{\rm z}(x,y)\right]. \tag{4}$$

The general solution of (the dimensionless) Eq.(4) can be written as [8],

$$A_{\rm z}(x,y) = \ln\left[\ 2|g'(\zeta)|\ /\ (1+|g(\zeta)|^2)\ \right], \tag{5}$$

where $g(\zeta)$ is a holomorphic function of the complex variable $\zeta = x + iy$, a prime denotes differentiation with respect to ζ.

[8] J. Liouville, in *Journal de Liouville* , Vol. XVIII, 71 (1853)

The Harris pinch is recovered by taking $g(\zeta) = \exp(\zeta)$.
The corresponding cylindrical solution, the Bennet pinch[9], is obtained with $g(\zeta) = \zeta/2$.

Additional physically relevant solutions[10] can be obtained with conformal transformations. All these solutions are locally equivalent to the Harris solutions.

This allows us e.g., to obtain the linear eigenfunctions for the perturbed vector potential $\delta A_{\mathrm{z}}(x, y)$ in the quasistatic approximation (i.e., in the so called "external region" outside the reconnection region) in any of these configurations directly, by transforming the well known tearing-mode eigenfunction for the Harris pinch.

[9] D. Bennet, Phys. Rev. **45**, 90 (1934).
[10] F. Ceccherini, C. Montagna, F. Pegoraro, G. Cicogna: Phys. Plasmas, **12**, 052506 (2005).

This result provides a convenient starting point e.g., for the study of reconnection instabilities and island coalescence in a configuration with a chain of magnetic islands, as described by the so-called periodic Schmid-Burgk-Fadeev sheet pinch[11]

$$A_{\rm z}(x,y) = -\ln\left[(1+\alpha^2)^{1/2}\cosh(x) + \alpha\cos(y)\right], \tag{6}$$

with $\alpha = \text{real constant}$ which follows from Eq.(5) with

$$g(\zeta) = [\alpha + (1+\alpha^2)^{1/2}\exp(\zeta)].$$

[11] J. Schmid-Burgk, Max Planck-Institut für Physik und Astrophysik, Report No. MPJ-PAF/P1, 3/65
V.M. Fadeev, I.F. Kvartskhava, N.N. Komarov, *Nucl. Fus.*, **5**, 202 (1965).

Equilibria with non uniform temperatures

Obtaining Vlasov equilibria with nonuniform temperatures poses some conceptual and technical problems even in the 1-D case.
From the technical point of view it turns out that imposing the neutrality condition is algebraically more involved[12] than in the isothermal case unless, for the sake of simplicity, we assume a cold ion distribution $f_i(\mathbf{x},\mathbf{v}) = n_e(\mathbf{x})\,\delta^3(\mathbf{v})$, where the particle position is now a constant of the motion. Such an ion distribution provides charge neutrality but does not contribute to the plasma current.
From the conceptual point of view we are confronted by a wide variety of physically viable functional dependence of the electron distribution function on the particle energy ϵ and canonical momentum p_{z}.

[12] Actually local neutrality need not be imposed in order to find physically interesting equilibria, but the inclusion of the electrostatic potential leads to a nonlinear system where Poisson's and Ampère's equations are coupled.

A guideline can be set by looking for solutions where the entropy minimization is performed at fixed canonical momentum in which case the Lagrange multipliers related to the particle and energy conservation become functions of the canonical momentum. On this basis we consider electron distribution functions of the form

$$F_e(\epsilon, p_\mathrm{z}) = \frac{n_0 g^2(p_\mathrm{z})}{(2\pi T_0/m_e)^{3/2}} \, \exp\left[-h^2(p_\mathrm{z})\ \epsilon/T_0\right] , \tag{7}$$

with h and g arbitrary functions. Integrating Eq.(7) over $v_\mathrm{x,y}$, we obtain

$$\begin{aligned}\mathcal{F}_e(v_\mathrm{z}^2, p_\mathrm{z}) \equiv \int\int dv_\mathrm{x}\ dv_\mathrm{y}\ F_e(\epsilon, p_\mathrm{z}) = \\ = \left[n_0 l(p_\mathrm{z})/(2\pi T_0/m_e)^{1/2}\right]\ \exp\left[-h^2(p_\mathrm{z})\ m_e v_\mathrm{z}^2/(2T_0)\right] ,\end{aligned}$$

with $l(p_\mathrm{z}) \equiv g^2(p_\mathrm{z})/h^2(p_\mathrm{z})$.

Macroscopic force balance

In the one dimensional case Ampère's equation can be interpreted as a dynamic problem for a "particle" with motion described by $A_z(x)$

$$\frac{d^2 A_z}{dx^2} = -j_{ez}(A_z), \tag{8}$$

where lengths are normalized on the electron skin depth $\lambda_e = c/\omega_{pe}$ and current densities on $en_0 v_{the}$.

In Eq.(8) the coordinate x plays the role of time, the force term is given by the (negative of the) current density and the effective potential $V(A_z)$ by its primitive. The effective potential has indeed a non trivial physical meaning, which can be used to search for solutions with the requested features.

In a time independent configuration the electron fluid momentum balance equation takes the form

$$\mathbf{J} \times \mathbf{B} = \nabla \cdot \mathbf{\Pi}, \tag{9}$$

where $\mathbf{\Pi} \equiv m_e \int d\mathbf{v}\ f_e(\mathbf{v} - \mathbf{V}_e)(\mathbf{v} - \mathbf{V}_e)$ is the pressure tensor,
$\mathbf{V}_e \equiv \int d\mathbf{v}\ f_e \mathbf{v} / \int d\mathbf{v}\ f_e = -\mathbf{j}_e / e n_e$ is the mean fluid velocity
and we have assumed that the term $(\mathbf{V}_e \cdot \nabla)\mathbf{V}_e$ vanishes.
In the one dimensional case the momentum balance reduces to

$$\frac{d}{dx}\left(\frac{\mathbf{B}^2}{2} + \Pi_{xx}\right) = 0. \tag{10}$$

This means that the effective potential $V(x)$ is represented by the xx component of the pressure tensor $\mathbf{\Pi}$, in fact combining (8) and (10) we obtain $V(A_z) - \Pi_{xx}$ is constant.

Quasi-constant density solutions

In a number of investigations on plasma stability it would be of interest to start from stationary Vlasov solutions with inhomogeneous temperature but uniform density, so as to separate the effects of temperature and density gradients.
Such configurations are not available within the class of solutions described by Eq.(7), unless we resort to a perturbative approach where the dependence of the electron distribution on $p_{\rm z}$ is assumed to be weak.

In this case if we set

$$l(p_{\rm z}) = h(p_{\rm z}) = 1 + \eta\, \tilde{h}(p_{\rm z}),$$

where $\eta > 0$ is a small parameter such that $\eta\, |\tilde{h}| \ll 1$ for all x and for all values of $v_{\rm z}$ within the main body of the distribution function, *the density is uniform up to $\mathcal{O}(\eta^2)$ terms* provided $\tilde{h}(p_{\rm z})$ is a first or second order polynomial in $p_{\rm z}$.

In the interesting case $\tilde{h}(p_\mathrm{z}) = p_\mathrm{z}^2$, to first order in η, we find

$$n_e/n_0 \sim 1 - \eta, \quad j_\mathrm{z}/(en_0 v_{the}) \sim -2\eta A_\mathrm{z}, \tag{11}$$

and

$$\Pi_\mathrm{xx}/(n_0 T_0) = \Pi_\mathrm{yy}/(n_0 T_0) \sim 1 - 2\eta(1 + A_\mathrm{z}^2), \quad \Pi_\mathrm{zz}/(n_0 T_0) \sim 1 - 2\eta(3 + A_\mathrm{z}^2),$$

where Π is the anisotropic pressure tensor.

Solving Ampère's equation we find an oscillatory magnetic field configuration with $A_\mathrm{z} = \cos\left[(2\eta)^{1/2} x\right]$ and an oscillatory pressure tensor.

Asymptotically constant solutions

We search for solutions with temperature and density distributions (and thus with magnetic field) that are *asymptotically constant*, and with density and temperature gradients near the null lines of the magnetic field. In particular we are interested in configurations where, in contrast to Harris solution, the *density at large x does not vanish*. It is worth noting that the systems described here are intrinsically *anisotropic* due to the asymmetric dependence on the three components of $\mathbf{v}$ of the distribution function. The only nonzero components of the pressure tensor are the diagonal ones, and they read

$$\Pi_{xx} = \Pi_{yy} = \frac{n_0 T_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dv_z \frac{g^2(p_z)}{h^4(p_z)} \mathrm{e}^{-h^2(p_z) v_z^2},$$

$$\Pi_{zz} = \frac{2 n_0 T_0}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dv_z \left(v_z - V_{ez}\right)^2 \frac{g^2(p_z)}{h^2(p_z)} \mathrm{e}^{-h^2(p_z) v_z^2}.$$

pegoraro@df.unipi.it

The difference between $\mathbf{\Pi}_{zz}$ and the other two components gives an estimate of the anisotropy of the system.
This property is important because the reconnection instabilities are strongly affected by anisotropy.

Moreover, Eq.(10) shows that the equilibrium rests only on the xx component of the pressure tensor, therefore we define the temperature as

$$T/T_0 = \mathbf{\Pi}_{xx}/n_e.$$

Another interesting feature is that these distribution functions can be subject to two- stream instabilities: in some cases in fact they can be double-peaked in v_z[13].

[13]This is also the problem in the case of the Harris distribution with an additional Maxwellian density-pedestal.

As a relevant example we consider the distribution function

$$F_e\left(p_z, \mathbf{v}^2\right) = [n_0/(\pi^{3/2} v_{the}^3)]\,\exp\left[-\left(\alpha + \tanh p_z\right)\mathbf{v}^2\right], \tag{12}$$

which is obtained with the choice $g\left(p_z\right) = 1,\ h^2\left(p_z\right) = \alpha + \tanh p_z,\ \alpha > 1$. This distribution function is single-peaked in v_z, as can be seen clearly from Fig.(2).

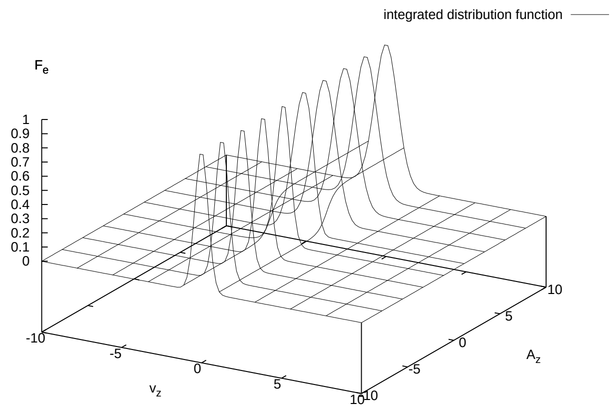


Figure 2: Distribution function in Eq.(12) integrated over v_x, v_y, for $\alpha = 2$.

The effective potential in which the "particle" that represents the vector potential moves has a hyperbolic tangent profile, as shown in Fig.(3).

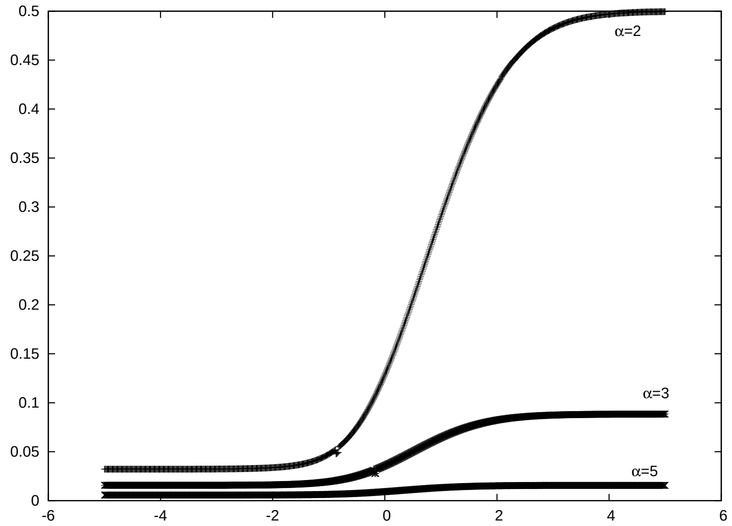


Figure 3: Π_{xx} versus A_z.

We choose $\alpha = 2$ and examine a "particle" that is reflected by the potential while coming from $-\infty$. We take $A_z(0) = -4, B_y(0) = -0.6$ as initial conditions.

The solution is plotted in Fig.(4), the x coordinate is chosen such that $B_y(0) = 0$.

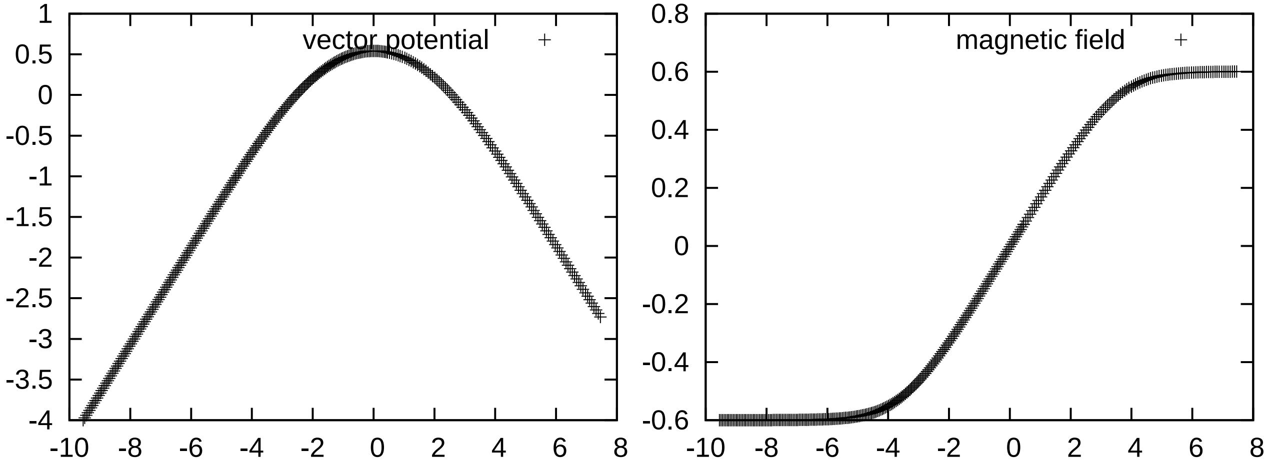


Figure 4: Vector potential and magnetic field.

The magnetic field has an hyperbolic tangent shape, resembling the Harris pinch configuration. As in Harris pinch the particle density, pressure and temperature have a maximum at $x = 0$, corresponding to the zero of the magnetic field, and are asymptotically constant (but not zero) at infinity.

The particle and current density, pressure and temperature profiles are shown in Fig.(5). The current sheet typical of the Harris pinch can be recognized.

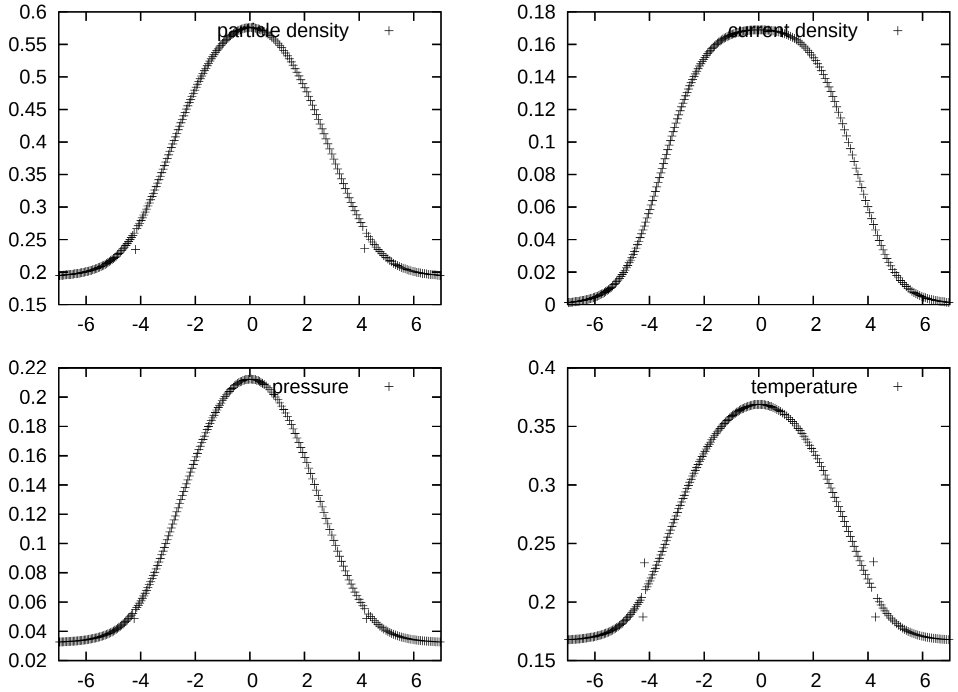


Figure 5: Particle and current densities, pressure and temperature profiles.

The anisotropy of the system is shown in in Fig.(6): Π_{xx} and Π_{zz} have almost the same behaviour, except that the latter is bigger.

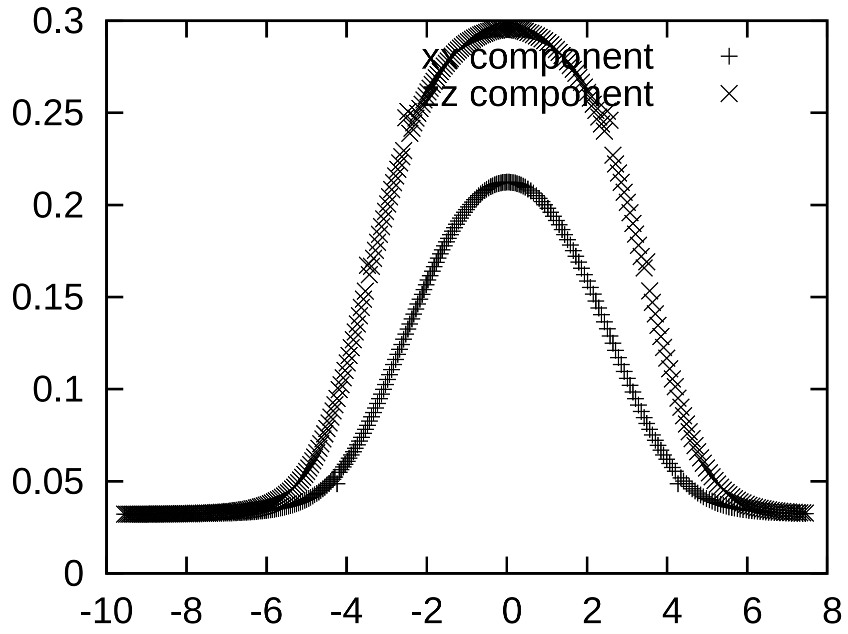


Figure 6: Diagonal components of the pressure tensor.

Conclusions

Stationary solutions of the Vlasov-Maxwell system in a magnetized inhomogeneous plasma have been described with the aim of illustrating easily obtainable models that can be adopted as the starting point of kinetic investigation of the stability and nonlinear evolution of magnetic configuration of interest for laboratory and for space plasmas.
In the *non isothermal* case explicit solutions have only been obtained in the one dimensional limit. A solution with properties similar to the Harris solution but with a non vanishing density at infinity has been presented. It has been shown that pressure anisotropy is an intrinsic feature of these solutions.
Along these same lines periodic equilibria can be constructed. Periodic equilibria have the advantage that the boundary conditions can be implemented more easily when simulating the dynamics of the system.